A Class of Effective Field Theory Models of Cosmic Acceleration

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Abstract

We explore a class of effective field theory models of cosmic acceleration involving a metric and a single scalar field. These models can be obtained by starting with a set of ultralight pseudo-Nambu-Goldstone bosons whose couplings to matter satisfy the weak equivalence principle, assuming that one boson is lighter than all the others, and integrating out the heavier fields. The result is a quintessence model with matter coupling, together with a series of correction terms in the action in a covariant derivative expansion, with specific scalings for the coefficients. After eliminating higher derivative terms and exploiting the field redefinition freedom, we show that the resulting theory contains nine independent free functions of the scalar field when truncated at four derivatives. This is in contrast to the four free functions found in similar theories of single-field inflation, where matter is not present. We discuss several different representations of the theory that can be obtained using the field redefinition freedom. For perturbations to the quintessence field today on subhorizon lengthscales larger than the Compton wavelength of the heavy fields, the theory is weakly coupled and natural in the sense of t'Hooft. The theory admits a regime where the perturbations become modestly nonlinear, but very strong nonlinearities lie outside its domain of validity.

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1 Introduction and Summary

1.1 Background and Motivation

The recent discovery of the accelerating expansion of the Universe [1, 2] has prompted many theoretical speculations about the underlying mechanism. The most likely mechanism is a cosmological constant, which is the simplest model and is in good agreement with observational data [3]. More complicated models involve new dynamical sources of gravity that act as dark energy, and/or modifications to general relativity on large scales. A plethora of models have been postulated and explored in recent years, including Quintessence, K-essence [4, 5], Ghost Condensates [6], DGP gravity [7], and $f(R)$ gravity, to name but a few. See Refs. [8, 9, 10, 11, 12, 13, 14] for detailed reviews of these and other models.

A common feature of the majority of dark energy and modified gravity models is that in the low energy limit, they are equivalent to general relativity coupled to one or more scalar fields, often called quintessence fields. Therefore it is useful to try to construct very general low energy effective quantum field theories of general relativity coupled to light scalar fields, in order to encompass broad classes of dark energy models. Considering dark energy models as quantum field theories is useful, even though the dynamics of dark energy is likely in a classical regime, because it facilitates discriminating against theories which are theoretically inconsistent or require fine tuning.

A similar situation occurs in the study of models of inflation, where it is useful to construct generic theories using effective field theory. Cheung *et al.* [15] constructed a general effective field theory for gravity and a single inflaton field, for perturbations about a background Friedman-Robertson-Walker cosmology in unitary gauge. This work was later generalized in multiple directions [16, 17] and has been very useful. An alternative approach to single field inflationary models was taken by Weinberg [18], who constructed an effective field theory to describe both the background cosmology and the perturbations. This theory consisted at leading order of a standard single field inflationary model with a potential, together with higher order terms in a covariant derivative expansion up to four derivatives. More detailed discussions of this type of effective field theory were given by Burgess, Lee and Trott [19].

When one turns from inflationary effective field theories to quintessence effective field theories, the essential physics is very similar, but there are three important differences that arise:

- First, the hierarchy of scales is vastly more extreme in quintessence models. The Hubble parameter H is typically several orders of magnitude below the Planck scale $m_p \sim 10^{28}$ eV in inflationary models, whereas for quintessence models $H_0 \sim 10^{-33}\,\text{eV}$ is ~ 60 orders of magnitude below the Planck scale. Quintessence fields must have a mass that is smaller than or on the order of H_0. It is a well-known, generic challenge for quintessence models to ensure that loop effects do not give rise to a mass much larger than H_0. Because of the disparity of scales, this issue is more extreme for quintessence models than inflationary models.

- In most inflationary models, it is assumed that the dynamics of the Universe are dominated by gravity and the scalar field (at least until reheating). By contrast, for quintessence

models in the regime of low redshifts relevant to observations, we know that cold dark matter gives an $O(1)$ contribution to the energy density. Therefore there are additional possible couplings and terms that must be included in an effective field theory.

- For any effective field theory, it is possible to pass outside the domain of validity of the theory even at energies E low compared to the theory's cutoff Λ, if the mode occupation numbers N are sufficiently large (see Sec. 5.2 below for more details). This corresponds to a breakdown of the classical derivative expansion. For quintessence theories, mode occupation numbers today can be as large as $N \sim (m_p/H_0)^2$ and it is possible to pass outside the domain of validity of the theory. By contrast in inflationary models, this is less likely to occur since mode occupation numbers for the perturbations are not large before modes exit the horizon. Thus, the effective field theory framework is less all-encompassing for quintessence models than for inflation models. This issue seems not to have been appreciated in the literature and we discuss it in Sec. 5.2 below.

Several studies have been made of generic effective field theories of dark energy. Creminelli, D'Amico, Noreña and Vernizzi [20] constructed a the general effective theory of single-field quintessence for perturbations about an arbitrary FRW background, paralleling the similar construction for inflation [15]. Park, Watson and Zurek constructed an effective theory for describing both the background cosmology and the perturbations, following the approach of Weinberg [18] but generalizing it to include couplings to matter [21].

The two approaches to effective field theories of quintessence – specialization to perturbations about a specific background, and maintaining covariance and the ability to describe the dynamics of a variety of backgrounds – are complementary to one another. The dynamics of the cosmological background FRW solution can be addressed in the covariant approach of Weinberg, but not in the background specific approach of Creminelli et al., which restricts attention to the dynamics of perturbations about a given, fixed background. On the other hand, a background specific approach can describe a larger set of dynamical theories for the perturbations than can a covariant derivative expansion[1].

1.2 Approach and Assumptions

The purpose of this paper is to revisit, generalize and correct slightly the covariant effective field theory analysis of Park, Watson and Zurek [21]. Following Weinberg and Park et al., we restrict attention to theories where the only dynamical degrees of freedom are a graviton and a single scalar. We allow couplings to an arbitrary matter sector, but we assume the validity of the weak equivalence principle, motivated by the strong experimental evidence for this principle. We assume that the theory consists of a standard quintessence theory coupled to matter at leading

[1]To see this, consider for example a term in the Lagrangian of the form $f(\phi)(\nabla\phi)^{2n}$, where ϕ is the quintessence field. Such a term would be omitted in the covariant derivative expansion for sufficiently large n. However, upon expanding this term using $\phi = \phi_0 + \delta\phi$, where ϕ_0 is the background solution, one finds terms $\sim (\nabla\phi_0)^{2n-2}(\nabla\delta\phi)^2$ which are included in the Creminelli approach of applying standard effective field theory methods to the perturbations.

order in a derivative expansion, with an action of the form

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = \int d^4x \sqrt{-g} \left\{ \frac{m_p^2}{2} R - \frac{1}{2} (\nabla\phi)^2 - U(\phi) \right\} + S_{\rm m} \left[e^{\alpha(\phi)} g_{\mu\nu}, \psi_{\rm m} \right]. \tag{1.1}$$

Here $\psi_{\rm m}$ denotes a set of matter fields, and m_p is the Planck mass. The factor $e^{\alpha(\phi)}$ in the matter action provides a leading-order non-minimal coupling of the quintessence field to matter, in a manner similar to Brans-Dicke models in the Einstein frame.

Our analysis then consists of a series of steps:

1. We add to the action all possible terms involving the scalar field and metric, in a covariant derivative expansion up to four derivatives. We truncate the expansion at four derivatives, as this is sufficient to yield the leading corrections to the action (1.1). As described by Weinberg [18] there are ten possible terms, with coefficients that can be arbitrary functions of ϕ [see Eq. (2.3) below]. Section 5.1 below describes one possible justification of this covariant derivative expansion from an effective field theory viewpoint, starting from a set of ultralight pseudo Nambu-Goldstone bosons (PNGBs). It is likely that the same expansion can be obtained from other, more general starting points.

2. We allow for corrections to the coupling to matter by adding to the metric that appears in the matter action all possible terms involving the metric and ϕ allowed by the derivative expansion, that is, up to two derivatives. There are six such terms [see Eq. (2.4) below.] We also add to the action terms involving the stress energy tensor $T_{\mu\nu}$ of the matter fields, up to the order allowed by the derivative expansion using $T_{\mu\nu} \sim m_p^2 G_{\mu\nu}$ [see Eq. (2.3) below]. Including such terms in the action seems poorly motivated, since *a priori* there is no reason to expect that the resulting theory would respect the weak equivalence principle (see Appendix B). However we show in Appendix B that the weak equivalence principle is actually satisfied, to the order we are working to in the derivative expansion. In addition, all the terms in the action involving $T_{\mu\nu}$ can be shown to have equivalent representations not involving the stress energy tensor, using field redefinitions (see Appendix B).

3. The various correction terms are not all independent because of the freedom to perform field redefinitions involving ϕ, $g_{\mu\nu}$ and the matter fields, again in a derivative expansion. In Sec. 3 we explore the space of such field redefinitions, finding eleven independent transformations and tabulating their effects on the coefficients in the action (see Table 1 below).

4. Several of the correction terms that are obtained from the derivative expansion are "higher derivative" terms, by which we mean that they give contributions to the equations of motion which involve third-order or higher-order time derivatives of the fields[2]. Normally,

[2]The precise definition of higher derivative that we use, which is covariant, is that an equation will be said not to contain any higher derivative terms if there exists a choice of foliation of spacetime for which any third-order or higher-order derivatives contain at most two time derivatives. Theories which are higher derivative in this sense are generically associated with instabilities (Ostragradski's theorem) [22], although the instabilities can be evaded in special cases, for example $f(R)$ gravity. For most of this paper (except for the Chern-Simons term), a simpler definition of higher derivative would be sufficient: a term in the action is "higher derivative" if it gives rise to terms in the equation of motion that involve any third-order or higher order derivatives.

such higher derivative terms give rise to additional degrees of freedom. However, if they are treated perturbatively (consistent with our derivative expansion) additional degrees of freedom do not arise. Specifically, one can perform a *reduction of order* procedure on the equations of motion [23, 24, 25], substituting the zeroth-order equations of motion into the higher derivative terms in the equations of motion to eliminate the higher derivatives[3]. We actually use a slightly different but equivalent procedure of eliminating the higher derivative terms directly in the action using field redefinitions[4] (see Appendix C).

Weinberg [18] and Park *et al.* [21] use a slightly different method, consisting of substituting the leading order equations of motion directly into the higher derivative terms in the action. This method is not generally valid, but it is valid up to field redefinitions that do not involve higher derivatives, and so it suffices for the purpose of attempting to classify general theories of dark energy (see Appendix C).

5. Another issue that arises with respect to the higher derivative terms is the following. Is it really necessary to include such terms in an action when trying to write down the most general theory of gravity and a scalar field, in a derivative expansion? Weinberg [18] suggested that perhaps a more general class of theories is generated by including these terms and performing a reduction of order procedure on them, rather than by omitting them. However, since it is ultimately possible to obtain a theory that is perturbatively equivalent to the higher derivative theory, and which has second order equations of motion, it should be possible just to write down the action for this reduced theory. In other words, an equivalent class of theories should be obtained simply by omitting all the higher derivative terms from the start. We show explicitly in Sec. 4 that this is the case for the class of theories considered here.

6. We fix the remaining field redefinition freedom by choosing a "gauge" in field space, thus fixing the action uniquely (see Sec. 4.2).

1.3 Results and Implications

Our final action is [Eq. (4.5) below]

$$
\begin{aligned}
S = &\int d^4x\sqrt{-g}\left\{\frac{m_p^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right\} + S_{\rm m}[e^{\alpha(\phi)}g_{\alpha\beta},\psi_{\rm m}] \\
&+ \epsilon\int d^4x\sqrt{-g}\Big\{a_1(\nabla\phi)^4 + b_2T(\nabla\phi)^2 + c_1G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\
&+ d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} + e_1T^{\mu\nu}T_{\mu\nu} + e_2T^2 + \ldots\Big\}.
\end{aligned}
\tag{1.2}
$$

[3]This is more general than requiring the solutions of the equation of motion to be analytic in the expansion parameter, as advocated by Simon [26]; see Ref. [25].

[4]This procedure is counterintuitive since normally field redefinitions do not change the physical content of a theory; here however they do because the field redefinitions themselves involve higher derivatives.

Here the coefficients a_1, b_2 etc. of the next-to-leading order terms in the derivative expansion are arbitrary functions of ϕ, and the ellipsis $\ldots$ refers to higher order terms with more than four derivatives. The corresponding equations of motion do not contain any higher derivative terms. This result generalizes that of Weinberg [18] to include couplings to matter.

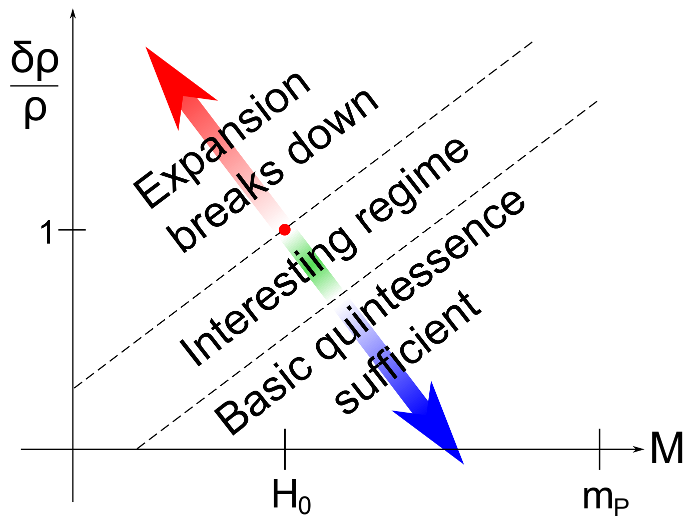

Figure 1: *The parameter space of fractional density perturbation $\delta\rho/\rho$ for perturbations to the quintessence field, and cutoff scale M for the effective field theory, illustrating the constraint (1.3) on the domain of validity. Near the boundary of the domain of validity the higher derivative terms in the action are potentially observable, this is labeled the "interesting regime". Further away from the boundary the higher derivative terms are negligible and the theory reduces to a standard quintessence model with a matter coupling.*

We can summarize our key results as follows:

- The most general action contains nine free functions of ϕ: $U, \alpha, a_1, b_2, c_1, d_3, d_4, e_1, e_2$, as compared to the four functions that are needed when matter is not present [18].

- There are a variety of different forms of the final theory that can be obtained using field redefinitions. In particular some of the matter-coupling terms in the action can be re-expressed as terms that involve only the quintessence field and metric. Specifically, the term $T(\nabla\phi)^2$ term could be eliminated in favor of $\Box\phi(\nabla\phi)^2$, the $(\nabla\phi)^4$ could be eliminated in favor of a term $T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, or the $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ term could be eliminated in favor of a term $T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ (see Sec. 4.2).

- As mentioned above, one obtains the correct final action if one excludes throughout the calculation all higher derivative terms.

- The final theory does contain terms involving the matter stress-energy tensor. Nevertheless, the weak equivalence principle is still satisfied (see Appendix B). It is possible to eliminate the stress-energy terms, but only if we allow higher derivative terms in the action (where it is assumed that the reduction of order procedure will be applied to these higher derivative terms). Thus, for a fully general theory, one must have either stress-energy terms or higher derivative terms; one cannot eliminate both (see Sec. 4.2).

- We can estimate how all the coefficients a_1 etc. scale with respect to a cutoff scale M for an effective field theory as follows (see Sec. 5.1). We assume that several ultralight scalar fields of mass $\sim H_0$ arise as pseudo Nambu-Goldstone bosons (PNGBs) from some high energy theory [27, 28], and are described by a nonlinear sigma model at low energies. We then suppose that all but one of the these PNGB fields have masses M that are somewhat larger than $\sim H_0$, and integrate them out. This will give rise to a theory of the form discussed above for the single light scalar, where the higher derivative terms are suppressed by powers of M. The scalings for each of the coefficients in the action are summarized in Table 3. We find that the fractional corrections to the cosmological dynamics due to the higher derivative terms scale as H_0^2/M^2, as one would expect.

- Finally, we can use these scalings to estimate the domain of validity of the effective field theory (see Sec. 5.2). We find that cosmological perturbations with a density perturbation $\delta\rho$ in the quintessence field must have a fractional density perturbation that satisfies

$$\frac{\delta\rho}{\rho} \ll \frac{M^2}{H_0^2}. \tag{1.3}$$

  Thus perturbations can become nonlinear, but only modestly so, if M is close to H_0. The parameter space of fractional density perturbation $\delta\rho/\rho$ and cutoff scale M is illustrated in Fig. 1. In addition there is the standard constraint for derivative expansions

$$E \ll M \tag{1.4}$$

  where E^{-1} is the length-scale or time-scale for some process. We show in Fig. 2 the two constraints (1.3) and (1.4) on the two dimensional parameter space of energy E and mode occupation number N.

Finally, in Appendix D we compare our analysis to that of Park, Watson and Zurek [21], who perform a similar computation but in the Jordan frame rather than the Einstein frame (see also Ref. [29]). The main difference between our analysis and theirs is that they use a different method to estimate the scalings of the coefficients, and as a result their final action differs from ours, being parameterized by three free functions rather than nine.

2 Class of Theories Involving Gravity and a Scalar Field

As discussed in the Introduction, our starting point is an action for a standard quintessence model with an arbitrary matter coupling, together with a perturbative correction which consists of a general derivative expansion up to four derivatives. The action is a functional of the Einstein-frame metric $g_{\alpha\beta}$, the quintessence field ϕ, and some matter fields which we denote collectively by $\psi_{\rm m}$:

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = S_0[g_{\alpha\beta}, \phi] + \epsilon S_1[g_{\alpha\beta}, \phi, T_{\alpha\beta}(\psi_{\rm m})] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}] + O(\epsilon^2). \tag{2.1}$$

Here $S_{\rm m}$ is the action for the matter fields, and the quantity ϵ is a formal expansion parameter. We will see in Sec. 5.1 below that ϵ can be identified as proportional to M^{-2}, where M is a

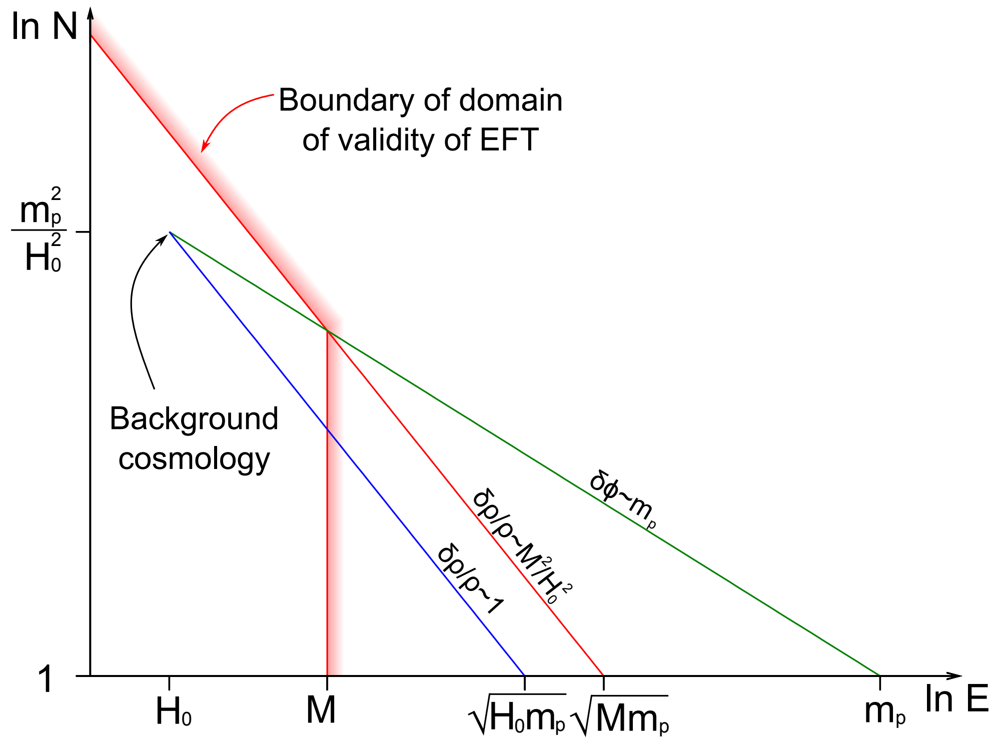

Figure 2: *The domain of validity of the effective field theory in the two dimensional parameter space of energy E per quantum of a mode of the quintessence field, and mode occupation number N. The cutoff scale M must be larger than the Hubble parameter H_0 in order that the background cosmology lie within the domain of validity. Perturbation modes on length-scales that are small compared to H_0^{-1} but large compared to M^{-1} can be described, but only if the mode occupation number and fractional density perturbation are sufficiently small. See Sec. 5.2 for details.*

cutoff scale or the mass of the lightest of the fields that have been integrated out to obtain the low energy action. Equivalently, ϵ counts the number of derivatives in our derivative expansion, with ϵ^n corresponding to $2n$ derivatives. The notation in the second term indicates that the perturbative correction S_1 to the action can depend on the matter fields, but only through their stress energy tensor $T_{\alpha\beta}$ (defined in Appendix A). Explicitly we have

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right], \tag{2.2}$$

and [18, 21]

$$\begin{aligned} S_1 = \int d^4x\sqrt{-g}\Big\{ & a_1(\nabla\phi)^4 + a_2\Box\phi(\nabla\phi)^2 + a_3(\Box\phi)^2 \\ & + b_1T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + b_2T(\nabla\phi)^2 + b_3T\Box\phi + b_4T^{\mu\nu}\nabla_\mu\nabla_\nu\phi + b_5R_{\mu\nu}T^{\mu\nu} \\ & + b_6RT + b_7T + c_1G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + c_2R(\nabla\phi)^2 + c_3R\Box\phi \\ & + d_1R^2 + d_2R^{\mu\nu}R_{\mu\nu} + d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) \\ & + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} + e_1T^{\mu\nu}T_{\mu\nu} + e_2T^2\Big\}. \end{aligned} \tag{2.3}$$

Here $C_{\mu\nu\lambda\rho}$ is the Weyl tensor and $\epsilon^{\mu\nu\lambda\rho}$ is the antisymmetric tensor (our conventions for these are given in Appendix A). There are additional terms with four derivatives that one can write down, but all such terms can be eliminated by integration by parts. Finally, the metric $\bar{g}_{\mu\nu}$ which appears in the matter action $S_{\rm m}$ in Eq. (2.1) is given by[5]

$$\begin{aligned}\bar{g}_{\mu\nu} =& e^{\alpha} g_{\mu\nu} + \epsilon e^{\alpha}\left[\beta_1 \nabla_\mu\phi\nabla_\nu\phi + \beta_2(\nabla\phi)^2 g_{\mu\nu} + \beta_3\Box\phi g_{\mu\nu} + \beta_4\nabla_\mu\nabla_\nu\phi + \beta_5 R_{\mu\nu} + \beta_6 R g_{\mu\nu}\right]\\ &+ O(\epsilon^2).\end{aligned} \tag{2.4}$$

All of the coefficients a_i, b_i, c_i, d_i, e_i, β_i and α are arbitrary functions of ϕ.

Let us briefly discuss each of the perturbative terms. The terms with coefficients a_i are corrections to the kinetic term of the scalar field. The b_i and β_i terms are couplings between the scalar field and the stress-energy tensor, or between curvature and the stress-energy tensor. The c_i terms are kinetic couplings between the scalar field and gravity. The d_i terms are quadratic curvature terms, which we have chosen to write as an R^2 term, an $R_{\mu\nu}R^{\mu\nu}$ term, and the Gauss-Bonnet term. Any constant piece of the coefficient d_3 is a topological term and may be omitted. The term d_4 is the gravitational Chern-Simons term, which may be excluded if one wishes to introduce parity as a symmetry of the theory, and again, any constant component of d_4 is topological and may be omitted. Finally, the e_i terms are quadratic in the stress-energy tensor.

Note that several of the terms in the action (2.3) are "higher derivative" terms, that is, they give rise to contributions to the equations of motion containing derivatives of order three or higher. The specific terms are those parameterized by the coefficients a_3, $b_3, \ldots, b_6$, c_2, c_3, d_1, d_2 and $\beta_3, \ldots, \beta_6$. As discussed in the Introduction and in Appendix C, we will choose to define our theory by treating these terms perturbatively, which excludes the extra degrees of freedom and instabilities that are normally associated with higher derivative terms.

We also note that the theory (2.1) satisfies the weak equivalence principle, to linear order in ϵ, as we show in Appendix B. That is, objects with negligible self-gravity with different compositions all experience the same acceleration. It is not *a priori* obvious that the principle should be satisfied since, as we show in Appendix B, violations of the principle generically arise whenever the matter stress energy tensor appears explicitly in the gravitational action, as in Eq. (2.1).

3 Transformation Properties of the Action

The description of the theory provided by Eqs. (2.1) – (2.4) is very redundant, in part because of the freedom to perform field redefinitions. In this section we derive how the various coefficients in the action (2.1) are modified under various transformations. In the next section we will use these transformation laws to derive a canonical representation of the theory, involving only nine free functions.

[5] We call this metric the Jordan frame metric, in an extension of the usual terminology which applies to the case when the relation (2.4) between the two metrics is just a conformal transformation.

3.1 Expansion of the Matter Action

Consider first the perturbative terms parameterized by $\beta_1, \ldots, \beta_6$, in the definition (2.4) of the Jordan metric $\bar{g}_{\alpha\beta}$, which appears in the matter action $S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]$. Using the definition (A.1) of the stress-energy tensor, we can eliminate these terms in favor of terms in the action involving $T_{\alpha\beta}$. Specifically we have from Eq. (A.1) that

$$S_{\rm m}[e^{\alpha}(g_{\mu\nu} + \delta g_{\mu\nu}), \psi_{\rm m}] = S_{\rm m}[e^{\alpha} g_{\mu\nu}, \psi_{\rm m}] + \frac{1}{2}\int d^4x\sqrt{-g}e^{2\alpha}T^{\mu\nu}\delta g_{\mu\nu} + O(\delta g^2). \tag{3.1}$$

Choosing

$$\delta g_{\mu\nu} = \epsilon[\tilde{\beta}_1 \nabla_\mu\phi\nabla_\nu\phi + \tilde{\beta}_2(\nabla\phi)^2 g_{\mu\nu} + \tilde{\beta}_3 \Box\phi g_{\mu\nu} + \tilde{\beta}_4 \nabla_\mu\nabla_\nu\phi + \tilde{\beta}_5 R_{\mu\nu} + \tilde{\beta}_6 R g_{\mu\nu}] \tag{3.2}$$

then gives a transformation of the action (2.1) characterized by the following changes in the coefficients:

$$\begin{array}{ll} \delta\beta_1 = -\tilde{\beta}_1, & \delta b_1 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_1, \\ \delta\beta_2 = -\tilde{\beta}_2, & \delta b_2 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_2, \\ \delta\beta_3 = -\tilde{\beta}_3, & \delta b_3 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_3, \\ \delta\beta_4 = -\tilde{\beta}_4, & \delta b_4 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_4, \\ \delta\beta_5 = -\tilde{\beta}_5, & \delta b_5 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_5, \\ \delta\beta_6 = -\tilde{\beta}_6, & \delta b_6 = \frac{1}{2}e^{2\alpha}\tilde{\beta}_6. \end{array} \tag{3.3}$$

Here the parameters $\tilde{\beta}_i$ can be arbitrary functions of ϕ. Similarly choosing $\delta g_{\mu\nu} = \epsilon\tilde{\alpha}g_{\mu\nu}$ gives a transformation characterized by

$$\delta\alpha = -\epsilon\tilde{\alpha}, \qquad \delta b_7 = \tfrac{1}{2}e^{2\alpha}\tilde{\alpha}. \tag{3.4}$$

3.2 Field Redefinitions Involving just the Scalar Field

Consider a perturbative field redefinition of the form

$$\phi = \psi + \epsilon\gamma, \tag{3.5}$$

where the quantity γ can in general depend on any of the fields and their derivatives. To leading order in ϵ, the change in the action (2.1) is then proportional to the zeroth-order equation of motion (5.10b) for ϕ. Relabeling ψ as ϕ, the change induced in the action is

$$\delta S = \epsilon\int d^4x\sqrt{-g}\gamma\left[\Box\phi - U' + \frac{1}{2}e^{2\alpha}\alpha' T\right]. \tag{3.6}$$

There are three special cases that will be useful:

1. First, choose

$$\phi = \psi + \epsilon\sigma_1 T, \tag{3.7}$$

where σ_1 is an arbitrary function[6] of ψ, and T is the trace of the stress-energy tensor. Substituting this into Eq. (3.6) and comparing with the general action (2.3), we find the following transformation law for the coefficients:

$$\begin{aligned} \delta b_3 &= \sigma_1, \qquad & \delta b_7 &= -U'\sigma_1, \\ \delta e_2 &= \tfrac{1}{2}\alpha' e^{2\alpha}\sigma_1. \end{aligned} \tag{3.8}$$

2. Second, we use the field redefinition

$$\phi = \psi + \epsilon\sigma_2[\Box\psi + U'(\psi)]. \tag{3.9}$$

Here the second term in the square bracket is included in order to maintain canonical normalization of the scalar field, that is, to avoid generating terms in the action of the form $f(\phi)(\nabla\phi)^2$. The resulting transformation law is

$$\begin{aligned} \delta a_3 &= \sigma_2, \qquad & \delta b_3 &= \tfrac{1}{2}e^{2\alpha}\alpha'\sigma_2, \\ \delta b_7 &= \tfrac{1}{2}\alpha' e^{2\alpha}U'\sigma_2, \qquad & \delta U &= \epsilon(U')^2\sigma_2. \end{aligned} \tag{3.10}$$

3. Third, consider the field redefinition

$$\phi = \psi + \epsilon\sigma_3 - \epsilon\sigma_3'(\nabla\psi)^2/U', \tag{3.11}$$

where σ_3 is a function of ψ and again the particular combination of terms is chosen to maintain canonical normalization. Substituting into Eq. (3.6), performing some integrations by parts and comparing with Eq. (2.3) gives the transformation law

$$\begin{aligned} \delta a_2 &= -\sigma_3'/U', \qquad & \delta b_2 &= -\tfrac{1}{2}e^{2\alpha}\alpha'\sigma_3'/U', \\ \delta b_7 &= \tfrac{1}{2}e^{2\alpha}\alpha'\sigma_3, \qquad & \delta U &= \epsilon U'\sigma_3. \end{aligned} \tag{3.12}$$

Note that this transformation is not well defined in general in the limit $U' \to 0$, because of the factors of $1/U'$. However, it is well defined in the limit $U' \to 0$, $\sigma_3' \to 0$ with σ_3'/U' kept constant.

3.3 Field Redefinitions Involving the Metric

We now consider a more general class of field redefinitions, where in addition to redefining the scalar field via Eq. (3.5), we also perturbatively redefine the metric via

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \epsilon F_{\alpha\beta}. \tag{3.13}$$

Here the quantity $F_{\alpha\beta}$ can depend on ψ, $\hat{g}_{\alpha\beta}$, their derivatives and the stress energy tensor. The corresponding change in the action is proportional to the equation of motion (5.10a). Relabeling $\hat{g}_{\alpha\beta}$ as $g_{\alpha\beta}$ and ψ as ϕ, the total change in the action is

$$\begin{aligned} \delta S =& \frac{\epsilon}{2}\int d^4x\sqrt{-g}F_{\alpha\beta}\left[-m_p^2 G^{\alpha\beta} + \nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}(\nabla\phi)^2 g^{\alpha\beta} - U g^{\alpha\beta} + e^{2\alpha}T^{\alpha\beta}\right] \\ &+ \epsilon\int d^4x\sqrt{-g}\gamma\left[\Box\phi - U' + \frac{1}{2}e^{2\alpha}\alpha' T\right]. \end{aligned} \tag{3.14}$$

[6] Because we are working to linear order in ϵ, it does not matter whether we take σ_1 to be a function of ϕ or of ψ.

Coeff.	Term	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
a_1	$(\nabla\phi)^4$						$\star$		$\star$	$\star$		
a_2	$\Box\phi(\nabla\phi)^2$			$\star$				$\star$		$\star$		
a_3 †	$(\Box\phi)^2$		$\star$									
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$								$\star$			$\star$
b_2	$T(\nabla\phi)^2$			$\star$			$\star$				$\star$	$\star$
b_3 †	$T\Box\phi$	$\star$	$\star$					$\star$				
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$									$\star$		
b_5 †	$R^{\mu\nu}T_{\mu\nu}$					$\star$						$\star$
b_6 †	RT				$\star$						$\star$	$\star$
b_7	T	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$					$\star$			$\star$	$\star$		
c_2 †	$R(\nabla\phi)^2$				$\star$		$\star$					
c_3 †	$R\Box\phi$							$\star$				
d_1 †	R^2				$\star$	$\star$						
d_2 †	$R^{\mu\nu}R_{\mu\nu}$					$\star$						
d_3	Gauss-Bonnet											
d_4	Chern-Simons											
e_1	$T^{\mu\nu}T_{\mu\nu}$											$\star$
e_2	T^2	$\star$									$\star$	
	U (potential)		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		

Table 1: This table shows which of the terms in our action (2.2) are affected by each of the eleven field redefinitions (3.7) – (3.29) that are parameterized by the functions $\sigma_1(\phi), \ldots, \sigma_{11}(\phi)$. The columns represent the redefinitions, and the rows represent terms. Daggers † in first column indicate "higher derivative" terms, that is, terms that give contributions to the equations of motion containing time derivatives of higher than second order. Stars $\star$ indicate that the coefficient of that row's term is altered by that column's field redefinition. We omit the coefficients α and $\beta_1, \ldots, \beta_6$ since those coefficients are degenerate with $b_1, \ldots, b_7$ by Eqs. (3.3) and (3.4).

Note that this formula includes the effect of the change in the Jordan frame metric (2.4) caused by the transformation (3.13). We now consider seven different transformations of this type:

4. The first case is a change to the metric proportional to $Rg_{\alpha\beta}$. In order to maintain canonical normalization of both the metric and the scalar field, that is, to avoid terms of the form $f(\phi)(\nabla\phi)^2$ and $f(\phi)R$, we need the following combination of terms in the field redefinition:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\sigma_4' \left(\frac{m_p^2}{U}\hat{R} + 4\right)\hat{g}_{\alpha\beta}, \tag{3.15a}$$

$$\phi = \psi + 4\epsilon\sigma_4, \tag{3.15b}$$

for some function $\sigma_4(\psi)$. Substituting into Eq. (3.14), performing some integrations by parts and comparing with Eq. (2.3) we obtain for the transformation law

$$\begin{array}{cc} \delta b_7 = 2e^{2\alpha}\alpha'\sigma_4 - 4e^{2\alpha}\sigma_4', & \delta c_2 = \frac{m_p^2}{U}\sigma_4', \\ \delta d_1 = -\frac{m_p^4}{U}\sigma_4', & \delta b_6 = -\frac{e^{2\alpha}}{U}m_p^2\sigma_4', \\ \delta U = 4\epsilon\left[U'\sigma_4 - 4U\sigma_4'\right]. & \end{array} \tag{3.16}$$

5. Next consider changes to the metric proportional to $R_{\alpha\beta}$. In order to maintain canonical normalizations we use the following combination of terms in the field redefinition:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta}(1 - 2\epsilon\sigma_5') - 2\epsilon\frac{m_p^2}{U}\sigma_5'\hat{R}_{\alpha\beta}, \tag{3.17a}$$

$$\phi = \psi + \epsilon\sigma_5, \tag{3.17b}$$

for some function $\sigma_5(\psi)$. This gives the transformation law

$$\begin{array}{cc} \delta b_7 = \frac{1}{2}e^{2\alpha}\alpha'\sigma_5 - e^{2\alpha}\sigma_5', & \delta c_1 = -\frac{m_p^2}{U}\sigma_5', \\ \delta d_1 = -\frac{m_p^4}{2U}\sigma_5', & \delta d_2 = \frac{m_p^4}{U}\sigma_5', \\ \delta b_5 = -\frac{m_p^2}{U}e^{2\alpha}\sigma_5', & \delta U = \epsilon\left[U'\sigma_5 - 4U\sigma_5'\right]. \end{array} \tag{3.18}$$

6. The next case is a change to the metric proportional to $(\nabla\phi)^2 g_{\alpha\beta}$. To maintain canonical normalization of the scalar field, we need in addition a change to the scalar field, with the combined transformation being

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\frac{\sigma_6'}{U}(\hat{\nabla}\psi)^2\hat{g}_{\alpha\beta}, \tag{3.19a}$$

$$\phi = \psi + 4\epsilon\sigma_6, \tag{3.19b}$$

for some function σ_6. The resulting transformation law for the coefficients is

$$\begin{array}{cc} \delta a_1 = \sigma_6'/U, & \delta b_2 = -e^{2\alpha}\sigma_6'/U, \\ \delta b_7 = 2e^{2\alpha}\alpha'\sigma_6, & \delta c_2 = -\sigma_6' m_p^2/U, \\ \delta U = 4\epsilon U'\sigma_6. & \end{array} \tag{3.20}$$

7. Next consider changes to the metric proportional to $\Box\phi g_{\alpha\beta}$. The required form of field redefinition that preserves canonical normalization of ϕ is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_7\hat{\Box}\psi\hat{g}_{\alpha\beta}, \tag{3.21a}$$
$$\phi = \psi + 4\epsilon U\sigma_7, \tag{3.21b}$$

for some function σ_7. The coefficients in the action then change according to

$$\begin{aligned} \delta a_2 &= -\sigma_7, & \delta b_3 &= e^{2\alpha}\sigma_7, \\ \delta b_7 &= 2e^{2\alpha}\alpha' U\sigma_7, & \delta c_3 &= m_p^2\sigma_7, \\ \delta U &= 4\epsilon UU'\sigma_7. \end{aligned} \tag{3.22}$$

8. The fifth case is a change to the metric proportional to $\nabla_\alpha\phi\nabla_\beta\phi$. The required form of field redefinition that preserves canonical normalization of ϕ is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\frac{\sigma_8'}{U}\hat{\nabla}_\alpha\psi\hat{\nabla}_\beta\psi, \tag{3.23a}$$
$$\phi = \psi + \epsilon\sigma_8, \tag{3.23b}$$

for some function σ_8. The coefficients in the action then change according to

$$\begin{aligned} \delta a_1 &= -\sigma_8'/(2U), & \delta b_1 &= -e^{2\alpha}\sigma_8'/U, \\ \delta b_7 &= \tfrac{1}{2}e^{2\alpha}\alpha'\sigma_8, & \delta c_1 &= m_p^2\sigma_8'/U, \\ \delta U &= \epsilon U'\sigma_8. \end{aligned} \tag{3.24}$$

9. Next consider a change in the metric proportional to $\nabla_\alpha\nabla_\beta\phi$. To preserve canonical normalization of ϕ we use the redefinitions

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_9\hat{\nabla}_\alpha\hat{\nabla}_\beta\psi, \tag{3.25a}$$
$$\phi = \psi + \epsilon U\sigma_9, \tag{3.25b}$$

for some function σ_8. The coefficients in the action then change according to

$$\begin{aligned} \delta a_1 &= -\tfrac{1}{2}\sigma_9', & \delta a_2 &= -\sigma_9, \\ \delta b_4 &= e^{2\alpha}\sigma_9, & \delta b_7 &= \tfrac{1}{2}e^{2\alpha}\alpha' U\sigma_9, \\ \delta c_1 &= m_p^2\sigma_9', & \delta U &= \epsilon UU'\sigma_9. \end{aligned} \tag{3.26}$$

10. A simple case is when the change in the metric is proportional to $Tg_{\alpha\beta}$, for which no change to the scalar field is required. The redefinition is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{10}T\hat{g}_{\alpha\beta}, \tag{3.27}$$

for some function σ_{10}. The transformation law for the coefficients is

$$\begin{aligned} \delta b_2 &= -\sigma_{10}, & \delta b_7 &= -4\sigma_{10}U, \\ \delta e_2 &= e^{2\alpha}\sigma_{10}, & \delta b_6 &= m_p^2\sigma_{10}. \end{aligned} \tag{3.28}$$

11. Similarly, no transformation to the scalar is required for the case of a change in the metric proportional to $T_{\alpha\beta}$. The redefinition is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{11}T_{\alpha\beta}, \tag{3.29}$$

for some function σ_{11}, and the corresponding transformation law is

$$\begin{aligned} \delta b_1 &= \sigma_{11}, & \delta b_2 &= -\tfrac{1}{2}\sigma_{11}, \\ \delta b_7 &= -\sigma_{11}U, & \delta e_1 &= e^{2\alpha}\sigma_{11}, \\ \delta b_5 &= -m_p^2\sigma_{11}, & \delta b_6 &= \tfrac{1}{2}m_p^2\sigma_{11}. \end{aligned} \tag{3.30}$$

The eleven[7] field redefinitions (3.7) – (3.29) are summarized in Table 1, which shows which coefficients are modified by which transformations.

4 Canonical Form of Action

In this section, we derive our final, reduced action (1.2) from the starting action (2.1), using the transformation laws derived in Sec. 3. There is some freedom in which terms we choose to eliminate and which terms we choose to retain. We choose to eliminate all terms that give higher derivatives in the equations of motion, so that the final theory is not a "higher derivative" theory. However, even after this has been accomplished, there is still some freedom in how the final theory is represented. We discuss this further in Sec. 4.2 below. The order of operations in the derivation is important, since we need to take care that terms which we have already set to zero are not reintroduced by subsequent transformations. Table 2 summarizes our calculations and their effects on the coefficients in the action at each stage in the computation.

4.1 Derivation

The steps in the derivation are as follows:

1. *Elimination of Derivative Terms in the Jordan Frame Metric:* The transformation (3.3) can be used to eliminate all of the terms involving derivatives in the Jordan frame metric (2.4), which are parameterized by the coefficients $\beta_1, \ldots, \beta_6$. This changes the coefficients of the terms in the action that depend linearly on the stress energy tensor, namely $b_1, \ldots, b_6$. As discussed in Appendix B, these terms involving the stress-energy tensor look like they might violate the weak equivalence principle, but in fact they do not.

2. *Elimination of Higher Derivative, Quadratic in Curvature Terms:* We next consider the terms in the action that are quadratic functions of curvature, whose coefficients are d_1, d_2, d_3 and d_4. The Chern-Simons term (d_4) and the Gauss-Bonnet term (d_3) give rise to well behaved equations of motion (in the sense that they not increase the number of degrees of freedom), so we do not attempt to eliminate these terms. By contrast, the terms

[7] We could also consider a twelfth redefinition given by $g_{\alpha\beta} = \hat{g}_{\alpha\beta}(1 - 2\epsilon\sigma_{12}')$, $\phi = \psi + \epsilon\sigma_{12} - \epsilon m_p^2\sigma_{12}'\hat{R}/U'$. However this redefinition is not independent of the first eleven; the same effect can be achieved by choosing $\sigma_1 = -e^{2\alpha}\sigma_{12}'/U'$, $\sigma_3 = -\sigma_{12}$, $\sigma_7 = \sigma_{12}'/U'$, $\sigma_{10} = e^{2\alpha}\alpha'\sigma_{12}'/(2U')$.

Step		1	2	3	3	4	5	6	6	7	Final
	Transformations	$\tilde{\beta}_j$	σ_4, σ_5	σ_9	σ_{10}, σ_{11}	σ_6, σ_7	σ_2, σ_3	σ_8	σ_1	$\tilde{\alpha}$	
Coeff.	Term in Action										
a_1	$(\nabla\phi)^4$			$\star$		$\star$		$\star$			$\checkmark$
a_2	$\Box\phi(\nabla\phi)^2$			$\star$		$\star$	$\to 0$				$\circ$
a_3 †	$(\Box\phi)^2$						$\to 0$				
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\star$			$\star$			$\to 0$			$\circ$
b_2	$T(\nabla\phi)^2$	$\star$			$\star$	$\star$	$\star$				$\checkmark$
b_3 †	$T\Box\phi$	$\star$				$\star$	$\star$		$\to 0$		
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$	$\star$		$\to 0$							
b_5 †	$R^{\mu\nu}T_{\mu\nu}$	$\star$	$\star$		$\to 0$						
b_6 †	RT	$\star$	$\star$		$\to 0$						
b_7	T		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\to 0$	
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$		$\star$	$\star$				$\star$			$\checkmark$
c_2 †	$R(\nabla\phi)^2$		$\star$			$\to 0$					
c_3 †	$R\Box\phi$					$\to 0$					
d_1 †	R^2		$\to 0$								
d_2 †	$R^{\mu\nu}R_{\mu\nu}$		$\to 0$								
d_3	Gauss-Bonnet										$\checkmark$
d_4	Chern-Simons										$\checkmark$
e_1	$T^{\mu\nu}T_{\mu\nu}$				$\star$						$\checkmark$
e_2	T^2				$\star$				$\star$		$\checkmark$
	U (potential)		$\star$	$\star$		$\star$	$\star$	$\star$			$\checkmark$
Coeff.	Term in $\bar{g}_{\mu\nu}$										
β_1	$\nabla_\mu\phi\nabla_\nu\phi$	$\to 0$									
β_2	$(\nabla\phi)^2 g_{\mu\nu}$	$\to 0$									
β_3 †	$\Box\phi g_{\mu\nu}$	$\to 0$									
β_4 †	$\nabla_\mu\nabla_\nu\phi$	$\to 0$									
β_5 †	$R_{\mu\nu}$	$\to 0$									
β_6 †	$Rg_{\mu\nu}$	$\to 0$									
	α (conf. factor)									$\star$	$\checkmark$

Table 2: This table shows the progression of manipulations we make in this section. The second column on the left lists the various terms in the action (2.3), or in the Jordan-frame metric (2.4). The first column lists the corresponding coefficients; daggers † indicate higher derivative terms. The numbers in the first row along the top refer to the numbered steps in the derivation in Sec. 4.1. The second row shows which transformation functions are used in each step. In the table, a star $\star$ indicates that the corresponding row's term receives a contribution from the corresponding column's reduction process, while $\to 0$ indicates that the term has been eliminated. The check marks $\checkmark$ in the last column indicate the remaining terms that are non-zero in the final action (4.5). Finally, circles $\circ$ in the last column indicate terms that are nonzero in alternative forms of the final action obtained using the transformations (3.11) or (3.23), as discussed in Sec. 4.2.

proportional to the squares of the Ricci scalar and Ricci tensor, parameterized by d_1 and d_2, do increase the number of degrees of freedom. We can eliminate these terms by using the transformations (3.15) and (3.17), with parameters chosen to be

$$\sigma_4 = \int d\phi \, \frac{U}{m_p^4}(d_1 + d_2/2), \qquad \sigma_5 = -\int d\phi \, \frac{U}{m_p^4} d_2. \tag{4.1}$$

These transformations will then modify the coefficients b_5, b_6, b_7, c_1 and c_2, as well as the potential U (see Table 1).

3. *Elimination of some of the Linear Stress-Energy Terms:* We next turn to terms which depend linearly on the stress-energy tensor, parameterized by $b_1, \ldots, b_6$. First, we can eliminate the term $b_4 T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ by using the transformation (3.25) with $\sigma_9 = -e^{-2\alpha}b_4$. This gives rise to changes in the coefficients a_1, a_2, b_7, c_1 as well as to the potential U. Second, we can eliminate the terms parameterized by b_5 and b_6 by using the transformations (3.27) and (3.29) with the parameters $\sigma_{10} = -(b_6 + b_5/2)/m_p^2$, $\sigma_{11} = b_5/m_p^2$. This changes the coefficients b_1, b_2, b_7, e_1 and e_2.

4. *Elimination of Kinetic Coupling of the Scalar to Curvature:* We next focus on the terms which kinetically couple the scalar field to gravity, namely $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, $R(\nabla\phi)^2$ and $R\Box\phi$. The first of these does not produce higher derivative terms in the equation of motion, so we focus on the remaining two terms, which are parameterized by c_2 and c_3. These terms can be eliminated using the transformations (3.19) and (3.21), with the parameters chosen to be

$$\sigma_6 = \int d\phi \, \frac{U}{m_p^2} c_2, \qquad \sigma_7 = -\frac{c_3}{m_p^2}. \tag{4.2}$$

These transformations then give rise to changes in the coefficients a_1, a_2, b_2, b_3, b_7 as well as to the potential U.

5. *Elimination of some of the Corrections to Scalar Field Kinetic Term:* Our action includes three corrections to the scalar kinetic term, parameterized by a_1, a_2 and a_3. Of these, only term a_3 contributes higher order derivatives to the equations of motion. We eliminate this term, and also the term a_2, by using the transformations (3.9) and (3.11) with

$$\sigma_2 = -a_3, \qquad \sigma_3 = \int d\phi \, U' a_2. \tag{4.3}$$

This gives rise to corrections to the coefficients b_2, b_3 and b_7 and to the potential U.

6. *Elimination of some Kinetic Couplings of the Scalar to Stress-Energy:* We next turn to the term $b_1 T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. We can eliminate this using the transformation (3.23) with the parameter choice

$$\sigma_8 = \int d\phi \, b_1 U e^{-2\alpha}. \tag{4.4}$$

This gives rise to changes in the coefficients a_1, b_7, c_1 and U, from Table 1. We can also eliminate the term $b_3 T\Box\phi$ by using the transformation (3.7) with $\sigma_1 = -b_3$. This changes the coefficients e_2 and b_7.

7. *Elimination of Trace of Stress-Energy Tensor Term:* The last step is to re-express the term $b_7 T$ in terms of an $O(\epsilon)$ correction to the conformal factor e^{α} by using the transformation (3.4) with $\tilde{\alpha} = -2e^{-2\alpha} b_7$.

4.2 Canonical Form of Action and Discussion

Applying the parameter specializations derived above to the action (2.1) we arrive at our final result:

$$
\begin{aligned}
S = & \int d^4x \sqrt{-g} \left\{ \frac{m_p^2}{2} R - \frac{1}{2} (\nabla\phi)^2 - U(\phi) \right\} + S_{\rm m}[e^{\alpha(\phi)} g_{\alpha\beta}, \psi_{\rm m}] \\
& + \epsilon \int d^4x \sqrt{-g} \Big\{ a_1 (\nabla\phi)^4 + b_2 T (\nabla\phi)^2 + c_1 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \\
& + d_3 \left(R^2 - 4 R^{\mu\nu} R_{\mu\nu} + R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right) + d_4 \epsilon^{\mu\nu\lambda\rho} C_{\mu\nu}{}^{\alpha\beta} C_{\lambda\rho\alpha\beta} + e_1 T^{\mu\nu} T_{\mu\nu} + e_2 T^2 \Big\}. \qquad (4.5)
\end{aligned}
$$

This action contains nine free functions of ϕ: $U, \alpha, a_1, b_2, c_1, d_3, d_4, e_1, e_2$. The corresponding equations of motion do not contain any higher derivative terms and are presented in Appendix E.

Our final result (4.5) can be re-expressed in a number of equivalent forms:

- First, the term $b_2 T (\nabla\phi)^2$ in the action can be eliminated in favor of a term proportional to $e^{2\alpha} \beta_2 (\nabla\phi)^2 g_{\mu\nu}$ in the Jordan frame metric (2.4) using the transformation (3.3). As discussed in Appendix B the latter representation makes explicit that the weak equivalence principle is satisfied.

- The term $b_2 T (\nabla\phi)^2$ could also be eliminated in favor of a term $a_2 \Box\phi (\nabla\phi)^2$, using the transformation (3.11) parameterized by σ_3, as long as $\alpha' \neq 0$[8]. The dynamics of a scalar quintessence field with kinetic terms of the latter type have recently been explored in detail in Ref. [30], who called the mixing of the scalar and metric kinetic terms in the equations of motion "kinetic braiding". The representation of this term as $a_2 \Box\phi (\nabla\phi)^2$ has some advantages for cosmological analyses: in this representation the dynamics of the term are confined to the scalar sector, while in the b_2 representation they are coupled to matter.

- The term $a_1 (\nabla\phi)^4$ can be eliminated in favor of a term $b_1 T^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$, using the transformation (3.23) parameterized by σ_8.

- Alternatively, the term $c_1 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ can be eliminated in favor of a term $b_1 T^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$, using the transformation (3.23) parameterized by σ_8. Our result in this representation agrees with that of Weinberg [18] when all the matter terms are dropped. The c_1 representation has the advantage over the b_1 representation that the corrections are confined to the scalar sector and do not involve matter. The term $c_1 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ has interesting effects: it can give rise to a self-tuning cosmology as well as potentially support a Vainshtein screening mechanism [31].

[8] More precisely the criterion is that the zeroth order term in the expansion in α' in powers of ϵ is nonzero. A nonzero α' that is proportional to ϵ would be insufficient to allow this transformation.

- As discussed in Appendix B, it is possible to eliminate all the stress-energy terms from the action by applying field redefinitions. This yields a form of the theory in which the weak equivalence principle is manifest. However, the resulting action contains higher derivative terms, unlike all the representations discussed so far in this subsection. As discussed in the Introduction and in Appendix C, to define the theory when higher derivative terms are present we use the reduction of order technique applied to the equations of motion.

- Finally, the result can be cast in the Jordan conformal frame by doing a conformal transformation, followed by some field redefinitions to simplify the answer. The result is similar in form to the Einstein frame action (4.5):

$$\begin{aligned} S[\tilde{g}_{\alpha\beta}, \tilde{\phi}, \psi_{\rm m}] = &\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}m_p^2 e^{-\alpha}\tilde{R} - \frac{1}{2}(\tilde{\nabla}\tilde{\phi})^2 - \tilde{U}(\tilde{\phi})\right] + S_{\rm m}[\tilde{g}_{\alpha\beta}, \psi_{\rm m}] \\ &+ \epsilon\int d^4x\sqrt{-g}\Big\{\tilde{a}_1(\tilde{\nabla}\tilde{\phi})^4 + \tilde{b}_2 T(\tilde{\nabla}\tilde{\phi})^2 + \tilde{c}_1\tilde{G}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} \\ &+ \tilde{d}_3\left(\tilde{R}^2 - 4\tilde{R}^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{R}_{\mu\nu\sigma\rho}\tilde{R}^{\mu\nu\sigma\rho}\right) + \tilde{d}_4\epsilon^{\mu\nu\lambda\rho}\tilde{C}_{\mu\nu}{}^{\alpha\beta}\tilde{C}_{\lambda\rho\alpha\beta} \\ &+ \tilde{e}_1 T^{\mu\nu}T_{\mu\nu} + \tilde{e}_2 T^2\Big\}. \end{aligned} \tag{4.6}$$

  Here $\tilde{g}_{\mu\nu} = e^{\alpha}g_{\mu\nu}$ and the field $\tilde{\phi}$ is a function of ϕ, where the function is chosen to give canonical normalization for $\tilde{\phi}$ in the Jordan frame action (4.6). All of the functions $\tilde{U}$, $\tilde{a}_1$, etc. in this action differ from the corresponding functions in the Einstein frame representation (4.5), but can in principle be computed in terms of them. The Jordan frame result (4.6) can also be cast in a number of different forms using linearized field redefinitions, just as for the Einstein frame result (4.5). Note that the stress energy tensor we use is the same in both frames, and is defined in Appendix A. The result (4.6) matches that found by Park *et al.* [21] (up to some minor adjustments, see Appendix D).

We note that the Chern-Simons term (d_4) gives rise to third-order derivatives in the equations of motion [see Eqs. (E.4) and (E.5) below]. However, with the choice of foliation[9] given by surfaces of constant ϕ, there are no third-order time derivatives, and so the Chern-Simons term is not a higher derivative term according to our definition (see the discussion in Sec. 1.2 above), and is not subject to the Ostrogradski instability. For further discussion of the Chern-Simons term in gravitational theories, see, e.g., Ref. [32]. As a parity-violating term, this term modifies the propagation speed of different polarizations of gravitons.

In the above derivation, we eliminated higher derivative terms using field redefinitions. As discussed in Appendix C, an alternative but equivalent procedure is to derive a form of the action which explicitly exhibits the extra degrees of freedom associated with the higher derivative terms, and then integrate out those degrees of freedom at tree level. This is shown explicitly for higher derivatives of the scalar field in Appendix C, and can also be shown explicitly for the terms d_1 and d_2 involving higher derivatives of the metric. A third, equivalent method is to perform a reduction

[9] This choice requires the assumption that $\nabla\phi$ is timelike everywhere, which will be true in cosmological applications when perturbations are sufficiently small.

of order procedure at the level of the equations of motion, as discussed in the Introduction and in Appendix C.

The above derivation confirms the general argument made in the Introduction that it should not be necessary to include higher derivative terms in the action. This is because the new terms that are generated when one eliminates the higher derivative terms should already be included in the derivative expansion. In the above derivation, if we eliminate from the start the higher derivative terms $(a_3, b_3, b_4, b_5, b_6, c_2, c_3, d_1, d_2)$, then we must also forbid all transformations that generate these terms, which includes all the transformations we have considered except Eqs. (3.3), (3.4), (3.11) and (3.23). The above derivation gets modified by dropping steps 2, 3, and 4, the portion of step 5 that sets a_3 to zero, and the portion of step 6 that sets b_3 to zero. The final result (4.5) is unchanged.

In a similar vein, the correct result can also be obtained by omitting from the initial action all terms involving the stress energy tensor, that is, the terms parameterized by $b_1, \ldots, b_7$ and e_1, e_2. If one follows all the steps of the derivation in Table 2, the same final result is obtained, and all the final coefficients are nonzero in general. This occurs because all the terms involving the stress energy tensor have alternative representations not involving it (although they do involve higher derivatives). Thus, from this point of view, it is not necessary to include in the action stress-energy terms.

However, it is *not* possible to do without both the higher derivative terms and the stress-energy terms. Suppose we throw out at the start all the higher derivative terms in both the action (2.1) and Jordan frame metric (2.4), and in addition omit all the stress-energy terms in the action. This would yield a version of the action (2.1) involving only the terms $a_1, a_2, \beta_1, \beta_2, c_1, d_3$ and d_4. Using the transformation (3.2) the terms β_1 and β_2 can be exchanged for b_1 and b_2, and the terms a_2 and b_1 can then be eliminated using the transformations parameterized by σ_3 and σ_8. This yields our final action (4.5) but without the terms e_1 and e_2. Therefore, for a fully general theory, one can choose to eliminate higher derivative terms, or stress-energy terms, but not both.

4.3 Extension to N scalar fields: Qui-N-tessence

The preceding analysis can be generalized straightforwardly to the case of N scalar fields, which we call "qui-N-tessence", an analog of multifield inflation [33, 16]. The zeroth order action (2.2) gets replaced by a general nonlinear sigma model:

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}q_{AB}(\phi^C)\nabla_\nu\phi^A\nabla_\mu\phi^B g^{\mu\nu} - U(\phi^C)\right], \tag{4.7}$$

where $\phi^A = (\phi^1, \ldots, \phi^N)$ are the N scalar fields and q_{AB} is a metric on the target space. In the remainder of the action, functions of ϕ are replaced by functions of ϕ^A. The first three terms of the second line of Eq. (4.5) are replaced by

$$\begin{aligned}&a_{1\,ABCD}\nabla_\mu\phi^A\nabla_\nu\phi^B\nabla_\lambda\phi^C\nabla_\sigma\phi^D g^{\mu\nu}g^{\lambda\sigma} + a_{2\,ABC}\nabla_\mu\nabla_\lambda\phi^A\nabla_\sigma\phi^B\nabla_\lambda\phi^C g^{\mu\nu}g^{\lambda\sigma}\\ &+ c_{1\,AB}G^{\mu\nu}\nabla_\mu\phi^A\nabla_\nu\phi^B.\end{aligned} \tag{4.8}$$

Thus the coefficients a_1, a_2 and c_1 become tensors on the target space of the indicated orders. Note that we must use the representation involving the coefficients $a_{1\,ABCD}, a_{2\,ABC}$ and not

$b_{1\,AB}, b_{2\,AB}$ (we assume $\alpha_{,A} \neq 0$) since the latter are less general; the equivalence between the different representations discussed in Sec. 4.2 does not generalize to the N field case. When $N \geq 4$ one could also add a term

$$a_{4\,ABCD} \nabla_\mu \phi^A \nabla_\nu \phi^B \nabla_\lambda \phi^C \nabla_\sigma \phi^D \epsilon^{\mu\nu\lambda\sigma}, \tag{4.9}$$

where $a_{4\,ABCD}$ is an arbitrary 4-form on the target space.

5 Order of Magnitude Estimates and Domain of Validity

In the previous sections, we started from the standard quintessence model with a matter coupling (2.2), and added arbitrary corrections involving the scalar field and metric in a derivative expansion up to four derivatives. We then exploited the field-redefinition freedom to eliminate all terms that give rise to additional degrees of freedom ("higher derivative terms"), and to reduce the set of operators in the action to the canonical and unique set given in our final action (4.5).

We now turn to estimating the scaling of the coefficients in the final action using effective field theory. We will then use these estimates to determine the domain of validity of the theory.

5.1 Derivation of Scaling of Coefficients

We start by recalling the scenario of pseudo Nambu-Goldstone Bosons (PNGBs) [27, 28] discussed in the Introduction that may give rise to the zeroth order action (2.2). Suppose that at some high energy scale M_* we spontaneously break a set of continuous global symmetries and thereby generate N massless Goldstone bosons $\phi^A = (\phi^1, \ldots, \phi^N)$. The theory then has N residual continuous symmetries. If we now suppose that these residual symmetries are explicitly broken at some much lower energy scale Λ, then a potential is generated that scales as

$$\Lambda^4 V(\phi^A/M_*), \tag{5.1}$$

for some function V which is of order unity. In particular the mass of the PNGB fields scale as Λ^2/M_* and can be very light. For example, in axion models $M_* \sim 10^{12}$ GeV and $\Lambda \sim \Lambda_{\rm QCD} \sim 100$ MeV, giving an axion mass of order 10^{-5} eV.

The leading order action for the PNGB fields coupled to gravity at low energies will be that of a nonlinear sigma model,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} m_p^2 R - \frac{1}{2} q_{AB}(\phi^C/M_*) \nabla_\nu \phi^A \nabla_\mu \phi^B g^{\mu\nu} - \Lambda^4 V(\phi^C/M_*) \right], \tag{5.2}$$

where q_{AB} is a metric on the target space which admits N Killing vectors (the residual symmetries). In the special case where q_{AB} is flat, these residual symmetries are shift symmetries $\phi^A \to \phi^A +$ constant. We now assume that these fields drive the cosmic acceleration, and in addition we assume that the kinetic and potential terms are of the same order, that is, we assume that slow roll parameters are only modestly small. It then follows from the action (5.2) that the scales of spontaneous and explicit symmetry breaking M_* and Λ must be of order[10]

$$M_* \sim m_p, \qquad \Lambda \sim \sqrt{H_0 m_p}, \tag{5.3}$$

[10]The need to use the Hubble scale today in the symmetry breaking scale Λ is associated with the coincidence problem.

where H_0 is the Hubble parameter, so that the quintessence fields have mass $\sim H_0$ and energy density $\sim m_p^2 H_0^2$. Defining the dimensionless fields $\varphi^A = \phi^A/m_p$ allows us to rewrite the action as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} m_p^2 R - \frac{1}{2} m_p^2 q_{AB}(\varphi^C) \nabla_\nu \varphi^A \nabla_\mu \varphi^B g^{\mu\nu} - m_p^2 H_0^2 V(\varphi^C) \right]. \tag{5.4}$$

Consider now the stability of the theory (5.4) under loop corrections. The story is exactly the same here as in inflationary models [34, 27] (aside from couplings to matter, see below). Computing loop corrections starting from the action (5.4) does not lead to large corrections $\delta m \gg H_0$ to the mass of the quintessence fields, because in the limit where the explicit symmetry breaking scale $\Lambda = \sqrt{m_p H_0}$ goes to zero, the theory possesses exact symmetries which must be respected by the loop corrections. Hence the loop corrections to the potential must scale proportionally to $H_0^2 m_p^2$, as for the original potential. Thus the smallness of the mass of the quintessence field is natural in the sense of 't Hooft. However, this is not the entire story, since the form (5.4) of the low energy theory imposes non trivial constraints on the physics at high energies, which must respect the residual symmetries. Indeed in general there is no guarantee that there exists a consistent high energy theory with the low energy limit (5.4). This question is beyond the scope of this paper: we shall simply assume that a consistent UV theory can be found. See Ref. [35] for an example of an attempt to address this issue.

So far in the discussion we have neglected coupling to matter. If we assume the validity of the weak equivalence principle, the general leading order coupling of ϕ^C to matter will be of the form of a scalar-tensor theory, given by adding to the action (5.4) the term

$$S_{\rm m}\left[e^{\alpha(\phi^c/M_*)} g_{\mu\nu}, \psi_{\rm m}\right] = S_{\rm m}\left[e^{\alpha(\varphi^c)} g_{\mu\nu}, \psi_{\rm m}\right], \tag{5.5}$$

for some function α.

We now suppose that one or more of the PNGB fields has a mass $\sim M$ which is parametrically larger than H_0, and we integrate out these heavier fields, following the similar analysis of inflationary models by Burgess, Lee and Trott [19]. Integrating out the heavier fields gives rise to modifications to the target space metric and potential for the remaining light fields [that do not change the scalings shown in Eq. (5.4)], and also a set of correction terms to the leading order action (5.4). The leading, tree-level correction terms can be obtained simply by solving the classical equations of motion for the heavy fields in an adiabatic approximation and substituting back into the action. One finds that the induced correction terms have the form[11]

$$M^2 m_p^2 \sum_n \frac{c_n}{M^d} \mathcal{O}_n, \tag{5.6}$$

where the sum is over operators $\mathcal{O}_n$ involving d derivatives acting on k powers of the dimensionless fields φ and/or $g_{\mu\nu}$, and the coefficients c_n are of order unity (see Appendix F for details). In other words, each additional derivative is suppressed by a power of the mass M of the fields that have been integrated out (which we can think of as a cutoff scale), and the overall prefactor is such that the normal kinetic terms would be reproduced for the case $k = d = 2$.

[11]These are the terms involving just the scalar field and metric. One also finds correction terms involving the matter stress energy tensor as long as $\alpha' \neq 0$, of the form indicated in Table 3.

Coefficient	Term in Action	Scaling
a_1	$(\nabla\phi)^4$	$\sim 1/(m_p^2 M^2)$
a_2	$\Box\phi(\nabla\phi)^2$	$\sim 1/(m_p M^2)$
a_3 †	$(\Box\phi)^2$	
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\sim 1/(m_p^2 M^2)$
b_2	$T(\nabla\phi)^2$	$\sim 1/(m_p^2 M^2)$
b_3 †	$T\Box\phi$	
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$	
b_5 †	$R^{\mu\nu}T_{\mu\nu}$	
b_6 †	RT	
b_7	T	
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\sim 1/M^2$
c_2 †	$R(\nabla\phi)^2$	
c_3 †	$R\Box\phi$	
d_1 †	R^2	
d_2 †	$R^{\mu\nu}R_{\mu\nu}$	
d_3	Gauss-Bonnet	$\sim m_p^2/M^2$
d_4	Chern-Simons	$\sim m_p^2/M^2$
e_1	$T^{\mu\nu}T_{\mu\nu}$	$\sim 1/(m_p^2 M^2)$
e_2	T^2	$\sim 1/(m_p^2 M^2)$

Table 3: This table gives the scalings of the various coefficients. The first column lists the coefficients, and the second column lists the corresponding terms in the action (2.3). Daggers in the first column indicate higher derivative terms. The third column gives our estimate of the scale of the coefficients, under the assumptions discussed in the text, for those coefficients that are nonzero in our final action (4.5), or in versions of that action obtained using the field redefinitions (3.11) or (3.23). The quantity M is the mass of the lightest field that is integrated out to produce our final action. In all cases, these scales for the coefficients correspond to fractional corrections to the leading order dynamics of order $\sim H_0^2/M^2$.

Note that the rule (5.6) for how the coefficients of additional corrections to the action depend on the cutoff scale M differs from the usual rule of effective field theory, where an operator of dimension $D + 4$ has a coefficient $\sim M^{-D}$. The rule (5.6) instead gives a coefficient $\sim M^{-(d-2)} m_p^{-(k-2)}$, where d is the number of derivatives in the operator and k is the number of powers of (canonically normalized) fields, related to D by $D = d + k - 4$. The difference between the two rules arises from the fact that we are making nontrivial assumptions about the physics above the scale M, specifically that it is described by an action of the PNGB form (5.4) [12]. If we were to allow arbitrary physics at energies above the scale M, then the coefficients would scale according to the standard rule.

We now specialize to the case of a single light field. The correction terms (5.6) have the form of a double power series, in number of derivatives and in powers of the fields. If we fix the number of derivatives and associated index structure, we can sum over all operators that differ only by powers of φ to obtain operators with prefactors that are functions of φ, $f(\varphi) = \sum c_k \varphi^k$ with coefficients of order unity. We now write out all the resulting terms to leading order in $1/M^2$, imposing general covariance. The result is the theory (4.5) discussed in the last section[13], but with additional information about the coefficients a_1, b_1 etc. Specifically we find that

$$a_1(\phi) = \frac{1}{m_p^2 M^2} \hat{a}_1(\phi/m_p), \tag{5.7}$$

where the function $\hat{a}_1$ is of order unity, i.e., the coefficients in its Taylor expansion are independent of m_p and M. The corresponding prefactors or overall scaling for the other coefficients are listed in Table 3.

Finally, we note that, as is well known, Solar System tests of general relativity strongly constrain the coupling of ϕ to the matter sector[14]. If we define the dimensionless parameter $\lambda = m_p |\alpha'(\phi_0)|$, where ϕ_0 is the present day cosmological background value of ϕ, then the Solar System constraint is[15] $\lambda \lesssim 10^{-2}$ [40]. In addition the coupling of the scalar to the visible sector will generically give rise to large corrections to the quintessence potential via loop corrections [41, 42, 43, 44, 45]. For a fermion of mass m_f, the correction δm to the mass of the quintessence field will be of order

$$\frac{\delta m}{H_0} \sim \lambda \left(\frac{m_f^2}{H_0 m_p} \right). \tag{5.8}$$

[12] More general interactions which are not of the form (5.4) can modify the scaling rule (5.6), even if they respect the residual (shift) symmetries. For example consider a scalar field ψ of mass m which couples to ϕ via a term $\psi(\nabla\phi)^2/m_*$ for some mass scale m_*. Integrating out this field gives a correction to the ϕ action $\sim (\nabla\phi)^4/(m^2 m_*^2)$ (see Appendix F). To keep such terms from invalidating the scaling rule we need to assume that $m m_* \gtrsim M m_p$, i.e. that any such fields are either sufficiently massive or sufficiently weakly coupled to the PNGB fields.

[13] The parity-violating Chern-Simons term is not generated in this way, since the fields we are integrating out do not violate parity. To obtain the Chern-Simons term with the scaling indicated in Table 3 would require integrating out some parity violating fields at the scale M which approximately respect the residual (shift) symmetries.

[14] Strictly speaking, Solar System tests lie outside the domain of validity of our effective field theory unless $M^{-1} \lesssim 1\,\text{A.U.}$, which is very small compared to H_0^{-1}; see Sec. 5.2 above.

[15] This constraint can be evaded in models where nonlinear effects in ϕ are important in the Solar System, such as Chameleon [36] and Galileon models [37, 38, 39].

If $\lambda \sim 1$ and $m_f \gg \sqrt{m_p H_0} \sim 10^{-3}$ eV, then $\delta m \gg H_0$, which is inconsistent if the quintessence field is to drive cosmic acceleration. This is a well known naturalness problem for matter couplings in quintessence models, and it motivates setting[16] $\alpha = 0$.

5.2 Domain of Validity of the Effective Field Theory

We now estimate the domain of validity for the theory (4.5) with the scalings given by Table 3, by requiring that the terms with higher derivatives be small compared to terms with fewer derivatives. If E is the energy involved in a given process, or equivalently E^{-1} is the corresponding time-scale or length-scale, then successive terms in the derivative expansion are suppressed by the ratio E/M, which yields the standard condition

$$E \ll M \tag{5.9}$$

for the domain of validity. As discussed in the Introduction, M must be somewhat larger than H_0 in order to describe the background cosmology and observable perturbation modes. However if M is significantly larger than H_0 then the corrections due to the higher order terms in Eq. (4.5) become negligible, and the theory reduces to a standard quintessence model with some matter coupling. Therefore, the interesting regime is when M is perhaps just one or two orders of magnitude larger than H_0, as indicated in Fig. 1. In particular, when the scale M is in this interesting regime, the theory is unable to describe gravitational effects in the Solar System and binary pulsars, which is a shortcoming of the effective field theory approach used here.

Consider now the background cosmological solution. The theory (2.1) to zeroth order in ϵ (or equivalently $1/M^2$) has the equations of motion

$$m_p^2 G_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(\nabla\phi)^2 g_{\alpha\beta} - U(\phi)g_{\alpha\beta} + e^{2\alpha(\phi)}T_{\alpha\beta}, \tag{5.10a}$$

$$\Box\phi = U'(\phi) - \frac{1}{2}\alpha' e^{2\alpha}T. \tag{5.10b}$$

For each of these two equations we assume that all of the terms are of the same order. For the matter terms this is this is a reasonable approximation, since $\Omega_\Lambda \sim 0.7$ and $\Omega_{\rm matter} \sim 0.3$. If the scalar potential term dominates over the kinetic term, then the following estimates need to be modified by including factors of slow roll parameters; we ignore these factors here since we expect them to be only modestly small. Similarly, our estimates assume that $m_p\alpha'$ is of order unity; some changes would be required if this quantity were very small. From these assumptions, and ignoring $O(1)$ functions of the scalar field, we have

$$m_p^2 R \sim (\nabla\phi)^2 \sim U \sim T \sim m_p\Box\phi \sim m_p U'(\phi) \sim H_0^2 m_p^2. \tag{5.11}$$

Inserting these estimates into the action (4.5) and using the scalings given in Table 3, we find that for each of the correction terms in the action, the fractional corrections to the leading order cosmological dynamics scale as H_0^2/M^2. The corrections therefore are of order unity at $M \sim H_0$, as we would expect, since at this scale the heavy fields which we have integrated out have the

[16]More precisely the condition is $\alpha' = 0$, i.e., $\alpha =$ constant, but the constant can be absorbed by a rescaling of all the dimensionful parameters in the matter action.

same mass scale as the light fields, and would be expected to give rise to $O(1)$ corrections to the dynamics. This gives a useful consistency check of the calculations underlying Table 3 discussed in the previous subsection.

In addition to the standard constraint (5.9), there are other constraints on the domain of validity which we now discuss. We focus attention on cosmological perturbations, for which $\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x})$, and consider the conditions under which the dynamics of the perturbation $\delta\phi$ can be described by the effective theory. Consider localized wavepacket modes $\delta\phi$, where the size of the wavepacket is of the same order as the wavelength, both $\sim E^{-1}$. For such modes we can characterize perturbations in terms of two parameters, the energy E and the number of quanta or mode occupation number N. The total energy of the wavepacket will be of order $NE \sim \int d^3x (\nabla\delta\phi)^2 \sim E^{-3}(E\delta\phi)^2$ which gives the estimate

$$\delta\phi \sim \sqrt{N}E. \tag{5.12}$$

The fractional density perturbation due to the wavepacket is of order

$$\frac{\delta\rho}{\rho} \sim \frac{(\nabla\delta\phi)^2}{H_0^2 m_p^2} \sim \frac{NE^4}{H_0^2 m_p^2}. \tag{5.13}$$

We now demand that the term $a_1(\nabla\delta\phi)^4$ in the action[17] be small compared to the leading order term $(\nabla\delta\phi)^2$. Using the scaling $a_1 \sim 1/(m_p^2 M^2)$ from Table 3 and combining with the estimate (5.13) of the fractional density perturbation then gives the constraint[18]

$$\frac{\delta\rho}{\rho} \ll \frac{M^2}{H_0^2}. \tag{5.14}$$

Thus, the theory can describe perturbations in the nonlinear regime, but the perturbations can only be modestly nonlinear if M is fairly close to H_0. In terms of the parameters E and N the constraint (5.14) is

$$NE^4 \ll M^2 m_p^2. \tag{5.15}$$

This gives a nontrivial constraint on the domain of validity of the theory in the regime $E \lesssim M$. The two dimensional parameter space (E, N) is illustrated in Fig. 2, which shows the constraints (5.9) and (5.14), the curves $\delta\rho/\rho \sim 1$ and $\delta\rho \sim M^2/H_0^2$, as well as the curve where $\delta\phi \sim m_p$.

Another potential constraint on the domain of the validity of the theory (4.5) with the scalings given by Table 3 is that the theory should be weakly coupled, i.e. the effects of loop corrections should be small. Using the power counting methods of Ref. [19] one can show that this is indeed

[17]Here we envisage computing an action for the perturbations by expanding the action (4.5) around the background cosmological solution, as in Ref. [19].

[18]In the previous subsection we showed that $a_1(\phi) = \hat{a}_1(\phi/m_p)/(M^2 m_p^2)$, where $\hat{a}_1$ is function for which all the Taylor expansion coefficients are of order unity. It follows that $\hat{a}_1 \sim 1$ for $\phi \sim m_p$. However the estimate (5.14) requires the stronger assumption $\hat{a}_1 \lesssim 1$ for $\phi \gg m_p$ which need not be valid. If we instead assume that $\hat{a}_1 \sim (\phi/m_p)^\alpha$ for $\phi \gg m_p$ then the constraint (5.15) gets replaced by $N(E/M)^\gamma \ll m_p^2/M^2$, where $\gamma = 2(4+\alpha)/(2+\alpha)$. This modifies the boundary of the domain of validity of the effective field theory shown in Fig. 2 by changing the slope of the tilted portion of the boundary. In the limit $\alpha \to \infty$ this portion of the boundary approaches the green curve $\delta\varphi \sim m_p$.

true within the domain $H_0 \lesssim E \ll M$ of interest. Strong coupling can arise due to tri-linear couplings, as discussed in Sec. 2.2 of Ref. [19], but this only occurs for energies far below the Hubble scale H_0, and so is not relevant to cosmological applications of the theory.

We note that there are several well known theories of cosmic acceleration that are not encompassed by our effective field theory. The form of our expansion requires that the dominant contribution to cosmic acceleration be the leading order scalar terms and not the higher order terms, and so theories in which other mechanisms provide the acceleration cannot be described in our formalism. One example is provided by K-essence models in which terms in the action like $(\nabla\phi)^4$, $(\nabla\phi)^6$ ... are all equally important. In particular this is true for ghost condensate models [6]. Also there are many cosmic acceleration models that exploit the Vainshtein effect [46, 47, 48] to evade Solar System constraints on light fields with gravitational-strength couplings. The Vainshtein effect relies on nonlinear derivative terms in the scalar field action. Although our class of theories includes models that demonstrate the Vainshtein mechanism, the mechanism only operates outside the domain of validity of our approach, as we require the nonlinear derivative terms to be small. The chameleon mechanism [36, 49], on the other hand, does not require nonlinearities in the derivatives of the scalar field, and thus may be analyzed in our formalism, although the regime in which a screening mechanism would be required to evade fifth force experiments and solar system constraints will be in the regime of validity of our analysis only for large enough values of the cutoff M.

6 Discussion and Conclusions

In this paper, we have investigated effective field theory models of cosmic acceleration involving a metric and a single scalar field. The set of theories we considered consists of a standard quintessence model with matter coupling, together with a general covariant derivative expansion, truncated at four derivatives. We showed that this class of theories can be obtained from a PNGB scenario, where one of the PNGB fields is lighter than all the others, and the heavier fields are integrated out. We showed that in constructing this class of theories, including higher derivative terms in the action, as suggested by Weinberg [18], does not give any increased generality. We also showed that complete generality requires one to include terms in the action that depend on the stress-energy tensor of the matter fields.

We now turn to a discussion of some of the advantages and shortcomings of the approach adopted here to describe models of dark energy. Some of the shortcomings are:

- By construction, our approach excludes theories where nonlinear kinetic terms in the action give an order unity contribution to the dynamics, such as K-essence, ghost condensates etc., since such theories do not arise from the PNGB construction used in this paper. On the other hand, such theories are less natural than the class of theories considered here, from the point of view of loop corrections: they require very nontrivial physics at the scale $\sim H_0$, instead of at the scale $\sim \sqrt{H_0 m_p}$ required in the PNGB approach. The most general class of theories of this kind is that of Horndeski [50], which contains four free functions of ϕ and $(\nabla\phi)^2$ [13], and which is the most general class of theories of a metric and a scalar field for which the equations of motion are second order. As discussed in the Introduction, these

theories are included in the alternative, background-dependent approach to effective field theories of quintessence of Creminelli et al. [20].

- Our class of theories will be observationally distinguishable from vanilla quintessence theories only if the cutoff M is near the Hubble scale H_0. In this regime, our framework cannot be used to analyze Solar System tests of general relativity, since they are outside the domain of validity of the effective field theory. Also, when the background cosmology is evolved backwards in time it passes outside the domain of validity at fairly low redshifts. (This is not a serious disadvantage since dark energy dominates only at low redshifts.)

- We have restricted attention to theories with a metric and a single scalar field, with the only symmetry being general covariance. Thus, our analysis does not include models with several scalar fields, vector fields etc. In addition, our analysis excludes an interesting class of models that one obtains by imposing that the action be invariant under $\phi \to f(\phi)$, where f is any monotonic function, as such a symmetry cannot be realized with our derivative expansion. This class of models includes Horava-Lifshitz gravity and has the same number of physical degrees of freedom as general relativity [13, 51]. It would be interesting to explore the most general dark energy models of this kind.

Some of the advantages of the approach used here are:

- Our class of theories is generic within the PNGB construction, which itself is a well motivated way to obtain the ultralight fields needed for cosmic acceleration. The theories are fairly simple and it should be straightforward to confront them with observational data.

- Our class of theories allow for a unified treatment of the cosmological background and perturbations, unlike the background-dependent approach of Ref. [20].

Finally, we list some possible directions in which the approach used here could be extended:

- It would be interesting to compute the relation between the nine free functions used in our theories to the free functions of the post-Friedmannian approach to parameterizing dark energy models [13].

- It would be interesting to explore the phenomenology of the various higher order terms in our action, for the cosmological background evolution and perturbations. Many of the terms have already been explored in detail, see for example Refs. [30, 31].

- Either by using the post-Friedmannian approach, or more directly, it would be useful to compute the current observational constraints on the free functions in the action.

- An interesting open question is the extent to which our final action is generic. That is, is there a class of theories more general than nonlinear sigma model PNGB theories for which our action is obtained by integrating out some of the fields?

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A Notation and Conventions

We use natural units with $c = \hbar = 1$, and define the the reduced Planck mass via $m_p^2 = 1/8\pi G$. The Einstein and Jordan frame metrics are $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ respectively, and the corresponding derivative operators are ∇_μ and $\bar{\nabla}_\mu$. We use the usual abbreviations $(\nabla\phi)^2 = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\psi$ and $\Box\phi = \nabla_\mu\nabla^\mu\phi$. Primes denote derivatives with respect to the scalar field ϕ, as in $U'(\phi)$. We use the $(-,+,+,+)$ metric signature and the sign conventions $(+,+,+)$ in the notation of Ref. [52]. Finally we take $\epsilon^{\mu\nu\lambda\rho}$ to be the antisymmetric tensor with $\epsilon^{0123} = 1/\sqrt{-g}$.

We define the (Jordan-frame) stress-energy tensor $T_\mu{}^\nu$ in the usual way in terms of the Jordan-frame metric $\bar{g}_{\mu\nu}$ that appears in the matter action $S_{\rm m}$:

$$S_{\rm m}[\bar{g}_{\mu\nu} + \delta\bar{g}_{\mu\nu}, \psi_{\rm m}] - S_{\rm m}[\bar{g}_{\mu\nu}, \psi_{\rm m}] = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}T_\mu{}^\nu \bar{g}^{\mu\lambda}\delta\bar{g}_{\lambda\nu} + O(\delta\bar{g}^2). \tag{A.1}$$

We then define $T = T_\mu{}^\mu$, and define the quantities $T_{\mu\nu}$ and $T^{\mu\nu}$ by raising and lowering indices with the Einstein-frame metric $g_{\mu\nu}$, which is related to $\bar{g}_{\mu\nu}$ via Eq. (2.4). To zeroth order in ϵ this stress energy tensor obeys the conservation law

$$e^{-2\alpha}\nabla_\lambda(e^{2\alpha}T^{\lambda\sigma}) = \frac{1}{2}\alpha' T\nabla^\sigma\phi + O(\epsilon). \tag{A.2}$$

B The Weak Equivalence Principle

In this Appendix, we show that including terms in the action that depend explicitly on the matter stress energy tensor, as in Eq. (2.1) above, generically gives rise to violations of the weak equivalence principle. However, we also show that our specific model (2.1) does not, to linear order in ϵ. Since the parameter ϵ essentially counts the number of derivatives in our derivative expansion, it follows the weak equivalence principle is satisfied for our derivative expansion up to four derivatives.

B.1 Generic Violations of Weak Equivalence Principle when Stress-Energy Terms are Present in Action

Consider first an action principle of the general form

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = S_{\rm g}[g_{\alpha\beta}, \phi] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]. \tag{B.1}$$

Here the first term is a gravitational action, depending only on the metric $g_{\alpha\beta}$ and the scalar field ϕ, and the second term is the matter action, in which all the matter fields $\psi_{\rm m}$ couple only to the Jordan metric $\bar{g}_{\alpha\beta}$ (some function of $g_{\alpha\beta}$ and ϕ), and not to $g_{\alpha\beta}$ and ϕ individually. By definition, any theory of this form obeys the weak equivalence principle. What this means is as follows. We define weakly self-gravitating bodies to be bodies for which we can neglect the perturbations they cause to $g_{\alpha\beta}$ and ϕ. From the form of the action (B.1), it follows that all weakly self-gravitating bodies will fall on geodesics of the metric $\bar{g}_{\alpha\beta}$, and hence will all fall on the same geodesics.

The action principle (2.1) we use in this paper is not of the general form (B.1), because of the explicit appearance of terms involving the stress energy tensor in the gravitational action.

Therefore one expects violation of the weak equivalence principle to arise. We now verify explicitly that this does occur in a specific example. We choose the following special case of the action (2.1), where the only perturbative term included is the term proportional to the trace of the stress energy tensor:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}m_p^2 R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon f(\phi) T\right] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]. \tag{B.2}$$

We choose the matter field $\psi_{\rm m}$ to be a scalar field ψ with action

$$S_{\rm m} = -\int d^4x\sqrt{-\bar{g}}\left[\frac{1}{2}(\bar{\nabla}\psi)^2 + V(\psi)\right], \tag{B.3}$$

and we specialize the relation (2.4) between the two metrics to be the conformal transformation $\bar{g}_{\alpha\beta} = e^{\alpha(\phi)}g_{\alpha\beta}$. This gives $T = -e^{-\alpha}(\nabla\psi)^2 - 4V$ and the total action is therefore

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}m_p^2 R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) - \frac{1}{2}(e^{\alpha} + 2\epsilon e^{-\alpha} f)(\nabla\psi)^2 - (e^{2\alpha} + 4\epsilon f)V(\psi)\right]. \tag{B.4}$$

The kinetic term for ψ can be written as $\int d^4x\sqrt{-\hat{g}}(\hat{\nabla}\psi)^2$ where $\hat{g}_{\alpha\beta} = (e^{\alpha} + 2\epsilon e^{-\alpha} f)g_{\alpha\beta}$, and the potential term can be written as $\int d^4x\sqrt{-\tilde{g}}V(\psi)$, where $\tilde{g}_{\alpha\beta} = \sqrt{e^{2\alpha} + 4\epsilon f}\, g_{\alpha\beta}$. Therefore, objects whose stress energy is composed of different combinations of the kinetic term and the potential term will fall on different combinations of the metrics $\hat{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$, violating the weak equivalence principle.

B.2 Validity of Weak Equivalence Principle to Linear Order

In the above analysis, we note that the metrics $\hat{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ coincide to linear order in ϵ, so there is no violation to this order. We now show that, similarly, none of the stress-energy-dependent terms included in Eq. (2.1) violate the weak equivalence principle, to linear order in ϵ.

The key idea of the proof is to use the transformation laws derived in Sec. 3 above to rewrite the theory in the general form (B.1), which we know satisfies the weak equivalence principle. All of the terms in the action given by Eqs. (2.1) – (2.4) are of this form, except for the terms parameterized by the coefficients $b_1, \ldots, b_7$, e_1 and e_2. However, as we now show, we can use transformations to eliminate these terms in favor of the remaining terms which manifestly satisfy the principle.

Consider first the terms in the action (2.3) which depend linearly on the stress-energy tensor. We can eliminate the terms parameterized by $b_1, \ldots, b_6$ using the transformation (3.3) with $\tilde{\beta}_i = -2e^{-2\alpha}b_i$ for $1 \le i \le 6$. This generates contributions to the the terms parameterized by $\beta_1, \ldots, \beta_6$ in the definition (2.4) of the Jordan metric. Similarly, by using the transformation (3.4) with $\tilde{\alpha} = -2e^{-2\alpha}b_7$, we can eliminate the term parameterized by b_7 in favor of an $O(\epsilon)$ correction to the function α in Eq. (2.4).

We now turn to the terms in the action (2.3) which depend quadratically on the stress-energy tensor, namely the terms parameterized by e_1 and e_2. For e_1 we use the transformation (3.29) with $\sigma_{11} = -e^{-2\alpha}e_1$, and for e_2 we use the transformation (3.27) with $\sigma_{10} = -e^{-2\alpha}e_2$. These transformations generates new contributions to the linear stress-energy terms parameterized by

b_1, b_2, b_5, b_6 and b_7 (see Table 1), but we have already shown that all of those terms satisfy the weak equivalence principle.

To summarize, we have shown that our model (2.1) satisfies the weak equivalence principle despite the explicit appearance of stress energy terms in the action. Of course, there can be violations of the strong equivalence principle in models of this kind, which can even be of order unity [53]. In addition, the weak equivalence principle will generically be violated by quantum loop corrections, although this is a small effect [54].

B.3 Potential Ambiguity in Definition of Weak Equivalence Principle

We next discuss a potential ambiguity that arises in the definition of the weak equivalence principle. In the definition one restricts attention to bodies whose gravitational fields, as measured by the perturbations they produce to the metric $g_{\mu\nu}$ and scalar field ϕ, can be neglected. However, consider for example the field redefinition (3.27), where the metric transforms according to

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{10}T\hat{g}_{\alpha\beta}. \tag{B.5}$$

It is possible for the perturbation $\delta\hat{g}_{\alpha\beta}$ generated by the body to be negligible, but the perturbation $\delta g_{\alpha\beta}$ to be non-negligible, because of the appearance of the stress-energy term in Eq. (B.5). If this occurs then the weak equivalence principle could be valid for one choice of variables, but not valid for the other choice.

To assess this ambiguity, we now make some order of magnitude estimates. Consider a body of mass $\sim M_b$ and size $\sim R$. Then in general relativity the size of the metric perturbation due to the body is of order $\delta\hat{g}_{\alpha\beta} \sim M_b/(m_p^2 R)$. Suppose now that $\sigma_{10} \sim 1/(m_p^2 M^2)$, as indicated by Eq. (3.28) and Table 3. Then the contribution to the metric perturbation $\delta g_{\alpha\beta}$ from the second term in Eq. (B.5) will be of order $M_b/(R^3 m_p^2 M^2)$, which will be much larger than $\delta\hat{g}_{\alpha\beta}$ whenever $R \ll M^{-1}$. Therefore the ambiguity could in principle arise.

However, in the models considered in this paper the ambiguity does not occur. This is because the condition $R \ll M^{-1}$ is excluded by the condition (5.9) for the validity of the effective field theory.

C Equivalence Between Field Redefinitions, Integrating Out New Degrees of Freedom, and Reduction of Order

The action (2.1) we start with in the body of the paper contains several higher derivative terms, that is, terms which gives contributions to the equations of motion which involve third-order and fourth-order time derivatives of the fields. As discussed in the Introduction, the theory with these higher derivative terms contains additional degrees of freedom compared to our zeroth order action (2.2), which contains a single graviton and scalar. In this paper our goal is to describe a general class of theories containing just one tensor and one scalar degree of freedom, so we wish to exclude these additional degrees of freedom[19].

[19] Higher derivative terms are also generically associated with instabilities [22], although this can be evaded in special cases, for example R^2 terms.

Therefore, as discussed in the Introduction, we define the theory we wish to consider, associated with our action (2.1), to be that obtained from the following series of steps:

1. Vary the action to obtain the equations of motion, which will contain third-order and fourth-order derivative terms which are proportional to ϵ.

2. Perform a *reduction of order* procedure on the equations of motion [23, 24, 25]. That is, substitute the zeroth-order in ϵ equations of motion into the higher derivative terms in order to obtain equations that contain only second-order and lower order time derivatives, which are equivalent to the original equations up to correction terms of $O(\epsilon^2)$ which we neglect.

3. Optionally, one can then derive the action principle that gives the reduced-order equations of motion.

In this Appendix, we show that this procedure is equivalent to the computational procedure we use in the body of the paper, in which we apply perturbative field redefinitions directly to the action in order to obtain an action with no higher derivative terms. We also show that it is equivalent to integrating out at tree level the extra degrees of freedom that are associated with the higher derivative terms.

We note that the analyses of general quintessence models by Weinberg [18] and Park *et al.* [21] used a different method of eliminating higher derivative terms. They performed a reduction of order procedure directly at the level of the action, that is, they substituted the zeroth-order equations of motion directly into the higher derivative terms in the action, to obtain an action with no higher derivative terms. We will show that this method is not in general correct; it does not agree with the theory obtained by applying the reduction of order method to the equations of motion[20]. However, it differs from the correct result only by field redefinitions (that do not involve higher derivatives), and so for the purpose of attempting to classify general theories of quintessence, Weinberg's method is adequate.

C.1 Reduction of Order Method

We start by considering the case of just a scalar field; a more general argument valid for scalar and tensor fields will be given below. Consider a general action of the form

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, \Box\phi]\right\}, \tag{C.1}$$

where F is an arbitrary function. We introduce the notation $K = (\nabla\phi)^2$ and $L = \Box\phi$. We first show that applying the reduction of order procedure to the equations of motion (steps $1-3$ above) give rise to a theory of the form (C.1) but with $F(\phi, K, L)$ replaced by another function $\hat{F}(\phi, K, L)$, given by

$$\hat{F}(\phi, K, L) = F[\phi, K, U'(\phi)] + [L - U'(\phi)]F_{,L}[\phi, K, U'(\phi)]. \tag{C.2}$$

[20]The reason is that substituting the zeroth order equations of motion into the action gives an action which is correct off-shell to $O(\epsilon^0)$ and on-shell to $O(\epsilon)$, but it needs to be valid off-shell to $O(\epsilon)$.

To see this, we vary the action (C.1) to obtain the equation of motion

$$\Box\phi - U'(\phi) + \epsilon F_{,\phi} - 2\epsilon\nabla_\alpha(F_{,K}\nabla^\alpha\phi) + \epsilon\Box F_{,L} = 0. \tag{C.3}$$

We now make the field redefinition

$$\psi = \phi + \epsilon F_{,L}[\phi, (\nabla\phi)^2, \Box\phi]. \tag{C.4}$$

Rewriting the equation of motion (C.3) in terms of ψ yields

$$\Box\psi - U'(\psi) + \epsilon U''(\psi)F_{,L} + \epsilon F_{,\phi} - 2\epsilon\nabla_\alpha(F_{,K}\nabla^\alpha\psi) = O(\epsilon^2), \tag{C.5}$$

where the arguments of $F_{,\phi}, F_{,L}$ and $F_{,K}$ are now $[\psi, (\nabla\psi)^2, \Box\psi]$.

We now apply the reduction of order procedure to the equation of motion given by Eqs. (C.4) and (C.5), that is, we substitute in the zeroth order equation of motion $\Box\psi = U'(\psi)$. The field redefinition (C.4) gets replaced by the following field redefinition which does not involve higher derivatives:

$$\psi = \phi + \epsilon F_{,L}[\phi, (\nabla\phi)^2, U'(\phi)] + O(\epsilon^2). \tag{C.6}$$

The equation of motion (C.5) is unchanged, except that the arguments of $F_{,\phi}, F_{,L}$ and $F_{,K}$ are now $[\psi, (\nabla\psi)^2, U'(\psi)]$. This equation of motion can be obtained from the action

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\psi)^2 - U(\psi) + \epsilon F[\psi, (\nabla\psi)^2, U'(\psi)]\right\}. \tag{C.7}$$

Finally we rewrite this action in terms of ϕ using the change of variable (C.6). The result is

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, U'(\phi)] + \epsilon[\Box\phi - U'(\phi)]F_{,L}[\phi, (\nabla\phi)^2, U'(\phi)]\right\}. \tag{C.8}$$

Note that although this action contains second order derivatives, the corresponding equations of motion contain derivatives only up to second order, that is, the theory is no longer a "higher derivative" theory [30]. The final, reduced-order action (C.8) is of the form (C.2) claimed above.

The final result (C.8) shows explicitly that the method of reducing order directly in the action used in Refs. [18, 21] is not correct. Applying this procedure to the action (C.1) would yield the first three terms in the action (C.8), but not the fourth term.

C.2 Method of Integrating Out the Additional Fields

We next show that the same result (C.8) can be obtained by integrating out the new degrees of freedom that are associated with the higher derivative terms. Starting from the action (C.1), we introduce an auxiliary scalar field ψ and consider the action

$$S[\phi,\psi] = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, \psi] + \epsilon(\Box\phi - \psi)F_{,L}[\phi, (\nabla\psi)^2, \psi].\right\}, \tag{C.9}$$

The equation of motion for ψ from this action is $\psi = \Box\phi$, assuming $F_{,LL} \neq 0$, and substituting this back into the action (C.9) yields the action (C.1). Thus the two actions are equivalent classically.

We now proceed to integrate out the field ψ, at tree level, i.e., classically. The equation of motion for ϕ is $\psi = U'(\phi) + O(\epsilon)$, and substituting this back the action (C.9) gives the same result (C.8) as was obtained from the reduction of order method.

C.3 Field Redefinition Method

We next turn to a discussion of the method we use to eliminate higher derivative terms in the body of the paper, using perturbative field redefinitions. That method is not generally applicable, but when it can be used, it is equivalent to the method of reduction of order (steps 1-3 above), as we now show. We start with an action of the form (C.1), with the function F chosen to be of the form

$$F(\phi, K, L) = g(\phi, K) + [L - U'(\phi)]h(\phi, K, L), \tag{C.10}$$

for some functions g and h. This is the most general form of F for which the field redefinition method can be used to eliminate the higher derivatives, and is sufficiently general to encompass the cases used in the body of the paper. First, we apply the reduction of order method. Inserting the formula (C.10) into Eq. (C.2) shows that the reduced-order action is characterized by the function $\hat{F}$ given by

$$\hat{F}(\phi, K, L) = g(\phi, K) + [L - U'(\phi)]h[\phi, K, U'(\phi)]. \tag{C.11}$$

However, the same result is obtained by starting with the action given by Eqs. (C.1) and (C.10) and performing the field redefinition

$$\phi \to \phi + \epsilon h[\phi, (\nabla\phi)^2, U'(\phi)] - \epsilon h[\phi, (\nabla\phi)^2, \Box\phi]. \tag{C.12}$$

This shows the reduction of order and field redefinition methods are equivalent.

We now give a more general and abstract argument for the equivalence, valid for any field content. Suppose we have a theory containing higher derivative terms in the action, proportional to ϵ. Suppose that we can find a linearized field redefinition, involving higher derivatives, that has the effect of eliminating all higher derivative terms from the action. We can then consider this process in reverse: starting from a theory which is not higher derivative, by making a linearized field redefinition we obtain another theory which has higher derivative terms, proportional to ϵ. However, the change in the action induced by the field redefinition must be proportional to the equations of motion. Hence, these higher derivative terms will be eliminated by applying Weinberg's method of substituting the zeroth order equations of motion into the $O(\epsilon)$ terms in the action. As we have discussed, Weinberg's procedure is valid up to a field redefinition of the type (C.6) which does not change the differential order.

D Comparison with Previous Work

In this Appendix we compare our analysis and results to those of Park, Watson and Zurek [21], who perform a similar computation with similar motivation, but obtain a somewhat different final result [Eq. (1) of their paper]. The main differences that arise are:

- They work throughout in the Jordan frame, whereas we work in the Einstein frame. This is a minor difference which only affects the appearance of the computations and results, since it is always possible to translate from one frame to another.

- As discussed in the Introduction and in Appendix C, they use Weinberg's method of eliminating the higher derivative terms, consisting of substituting the zeroth order equations of motion into the higher derivative terms in the action, whereas we use the field redefinition method. The two methods are not equivalent for a given specific theory with specific coefficients, but are equivalent for the purpose of determining a general class of theories.

- After eliminating higher derivative terms, their result is an action [Eq. (5) of their paper] that contains eleven functions of the scalar field, whereas our corresponding result (4.6) has only nine free functions. However, this is a minor difference: their function $Z(\phi)$ can be eliminated by redefining the scalar field to attain canonical normalization, and their function $f(\phi)$ can be eliminated by the transformation used in step 7 in Sec. 4.1 above.

- Another minor difference is that in their analysis they have in their action a Weyl squared term $\propto C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$, which is unaffected by any of the transformation they make to the action. This Weyl squared term gives rise to higher derivative terms in the equation of motion that are associated with ghost-like additional degrees of freedom [55]. In our analysis the Weyl squared term is replaced by the Gauss-Bonnet term, which is not a higher derivative term, because it would be a topological term if it were not for the ϕ-dependent prefactor.

- Aside from the above minor differences, our result (4.5) is equivalent to the result given in Eq. (5) of their paper. Two major differences arise subsequently in the estimates of the scalings for the coefficients of the operators in the Lagrangian.

  First, Park *et al.* use the standard effective theory scaling rule wherein an operator of dimension $4+n$ has a coefficient $\sim \Lambda^{-n}$, where Λ is the cutoff. As discussed in Sec. 5.1 above, this corresponds to placing no restrictions on the theory that applies above the cutoff scale Λ. By contrast, our approach does place restrictions on the physics at scales above Λ, and yields the modified scaling rule (5.6). As a consequence, our cutoff Λ (which we denote by M in the body of the paper) can be taken all the way down to the Hubble scale $H_0 \sim 10^{-33}$ eV, whereas their cutoff must be larger than $\sim \sqrt{H_0 m_p} \sim 10^{-3}$ eV.

  Second, Park *et al.* actually assume separate cutoffs for the gravitational, matter and scalar sectors of the theory, and estimate how each of their coefficients scale as functions of these three cutoffs. We do not understand completely their method of derivation of these scalings, but we do note that some of their scaling estimates are inconsistent with how the coefficients transform into one another under field redefinitions as discussed in Sec. 3 above. They then

proceed to drop some terms which their scalings indicate are subdominant, and arrive at a final action [Eq. (1) in their paper] which differs from ours, being parameterized by three free functions rather than nine.

E Equations of Motion for Reduced Theory

In this Appendix we compute the equations of motion for our final action (4.5), with the e_1 and e_2 terms omitted. We start by using a transformation of the form (3.2) with $\tilde{\beta}_2 = -2e^{-2\alpha}b_2$. This yields the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{m_p^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) + a_1(\nabla\phi)^4 + c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \right.$$
$$\left. +d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta}\right\}$$
$$+ S_{\rm m}\left[e^{\alpha(\phi)}g_{\mu\nu}\left(1+\beta(\nabla\phi)^2\right), \psi_{\rm m}\right]. \tag{E.1}$$

Here we have defined $\beta = 2e^{-2\alpha}b_2$; this was denoted β_2 in the body of the paper. We have also set $\epsilon = 1$ for simplicity. The representation (E.1) is more convenient than (4.5) for computing the equations of motion since it avoids varying of the stress-energy tensor.

Next, we vary the matter action in Eq. (E.1) using the definition (A.1) of the stress energy tensor $T_{\mu\nu}$ and the definition (2.4) of the Jordan metric $\bar{g}_{\mu\nu}$. This yields

$$\delta S_m = -\frac{1}{2}\int d^4x\sqrt{-g}e^{2\alpha}\left\{\delta g^{\mu\nu}\left[T_{\mu\nu} + 2T_{\mu\nu}\beta(\nabla\phi)^2 - \beta T\nabla_\mu\phi\nabla_\nu\phi\right]\right.$$
$$\left. +\delta\phi\left[-\alpha' T + 2\alpha'\beta T(\nabla\phi)^2 + \beta' T(\nabla\phi)^2 + 2\beta\nabla_\mu T\nabla^\mu\phi + 2\beta T\Box\phi\right]\right\}. \tag{E.2}$$

Combining this with the variation of the gravitational action gives the equations of motion

$$\Box\phi = U'(\phi) - \frac{1}{2}e^{2\alpha}\alpha' T + 4a_1\left[(\nabla\phi)^2\Box\phi + 2\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi\right] + 3a_1'(\nabla\phi)^4 + c_1' G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$
$$+ 2c_1 G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - d_3'\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) - d_4'\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta}$$
$$+ \frac{1}{2}e^{2\alpha}\left[2\alpha'\beta T(\nabla\phi)^2 + \beta' T(\nabla\phi)^2 + 2\beta\nabla_\mu T\nabla^\mu\phi + 2\beta T\Box\phi\right], \tag{E.3}$$

and

$$m_p^2 G_{\mu\nu} = e^{2\alpha}T_{\mu\nu} + \nabla_\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}(\nabla\phi)^2 + U(\phi)\right]g_{\mu\nu} - 4a_1(\nabla\phi)^2\nabla_\mu\phi\nabla_\nu\phi + a_1(\nabla\phi)^4 g_{\mu\nu}$$
$$+ g_{\mu\nu}c_1 G^{\sigma\lambda}\nabla_\sigma\phi\nabla_\lambda\phi - 4c_1 R_{\sigma(\mu}\nabla_{\nu)}\phi\nabla^\sigma\phi + c_1(R_{\mu\nu}(\nabla\phi)^2 + R\nabla_\mu\phi\nabla_\nu\phi)$$
$$- g_{\mu\nu}\nabla_\sigma\nabla_\lambda(c_1\nabla^\sigma\phi\nabla^\lambda\phi) + g_{\mu\nu}\Box[c_1(\nabla\phi)^2] + 2\nabla_\lambda\nabla_{(\mu}(c_1\nabla_{\nu)}\phi\nabla^\lambda\phi) - \nabla_\mu\nabla_\nu[c_1(\nabla\phi)^2]$$
$$- \Box(c_1\nabla_\mu\phi\nabla_\nu\phi) + 2R\nabla_\mu\nabla_\nu d_3 - 2g_{\mu\nu}R\Box d_3 + 4R_{\mu\nu}\Box d_3 - 8R^\sigma{}_{(\mu}\nabla_{\nu)}\nabla_\sigma d_3$$
$$+ 4g_{\mu\nu}R_{\sigma\rho}\nabla^\sigma\nabla^\rho d_3 + 4R_{\rho\mu\nu\sigma}\nabla^\rho\nabla^\sigma d_3 + 16C_{\mu\nu} + 2e^{2\alpha}T_{\mu\nu}\beta(\nabla\phi)^2 - e^{2\alpha}\beta T\nabla_\mu\phi\nabla_\nu\phi. \tag{E.4}$$

Here the tensor $C_{\mu\nu}$ comes from the Chern-Simons term, and is defined by

$$C^{\mu\nu} = (\nabla_\sigma d_4)\epsilon^{\sigma\lambda\rho(\mu}\nabla_\rho R^{\nu)}{}_\lambda + (\nabla_\sigma\nabla_\lambda d_4)^\star R^{\lambda(\mu\nu)\sigma} \tag{E.5}$$

where ${}^\star R^{\mu\nu\sigma\lambda} = \epsilon^{\sigma\lambda\rho\tau}R^{\mu\nu}{}_{\rho\tau}/2$. Note that the zeroth order terms involving the stress-energy tensor depend implicitly on β through the expression for the Jordan metric given in Eq. (E.1).

The terms involving c_1 are written in the most compact manner we could find. Although it looks unlikely, the higher order derivatives in these terms do cancel; the full expansion of these terms is

$$\begin{aligned}
&2c_1 g_{\mu\nu}R^{\sigma\lambda}\nabla_\sigma\phi\nabla_\lambda\phi - \frac{1}{2}c_1 g_{\mu\nu}R(\nabla\phi)^2 - 4c_1 R_{\sigma(\mu}\nabla_{\nu)}\phi\nabla^\sigma\phi + c_1 R_{\mu\nu}(\nabla\phi)^2 + c_1 R\nabla_\mu\phi\nabla_\nu\phi \\
&+ g_{\mu\nu}\left[c_1'\nabla_\sigma\phi\nabla_\lambda\phi\nabla^\sigma\nabla^\lambda\phi + c_1\nabla_\sigma\nabla_\lambda\phi\nabla^\sigma\nabla^\lambda\phi - c_1'(\nabla\phi)^2\Box\phi - c_1(\Box\phi)^2\right] \\
&- 2c_1\nabla_\sigma\nabla_\mu\phi\nabla^\sigma\nabla_\nu\phi - 2c_1'\nabla_\sigma\phi\nabla_{(\mu}\phi\nabla_{\nu)}\nabla^\sigma\phi + c_1'\nabla_\mu\nabla_\nu\phi(\nabla\phi)^2 + c_1'\nabla_\mu\phi\nabla_\nu\phi\Box\phi \\
&+ 2c_1\nabla_\mu\nabla_\nu\phi\Box\phi + 2c_1\nabla^\lambda\phi\nabla^\sigma\phi R_{\sigma\mu\nu\lambda}.
\end{aligned} \tag{E.6}$$

F Scaling of Coefficients Obtained by Integrating Out Pseudo-Nambu-Goldstone Fields

In this Appendix we give some more details of the derivation discussed in Sec. 5.1 of the scaling of the coefficients of the operators in the Lagrangian. We divide the PNGB fields Φ^A into two groups, a set χ^a with mass $\sim H_0$ and a set ψ^Γ with mass $\sim M$, where $M \gg H_0$:

$$\Phi^A = (\chi^a, \psi^\Gamma). \tag{F.1}$$

We assume an action for these fields of the form

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}q_{AB}(\Phi^A)\nabla_\mu\Phi^A\nabla_\nu\Phi^B g^{\mu\nu} - H_0^2 V\left(\chi^a, \frac{M}{H_0}\psi^\Gamma\right)\right\}. \tag{F.2}$$

This is the same as the action (5.4) of Sec. 5.1 above, except that an extra factor has been inserted into the potential to make the ψ^Γ fields have mass $\sim M$ rather than $\sim H_0$, and we have specialized to units where $m_p = 1$. We assume that the target space coordinates have been chosen so that the potential is minimized at $\psi^\Gamma = 0$, i.e.

$$V_{,\Gamma} = 0 \tag{F.3}$$

at $\psi^\Gamma = 0$.

We now want to let M become large and integrate out the fields ψ^Γ at tree level. This can be done by using Feynman diagrams and using power counting[21], as in Ref. [19]. Alternatively and more simply, it can be done by writing out the equations of motion for the fields ψ^Γ and invoking

[21] We note that Burgess *et al.* [19] write down a scaling rule in their Eqs. (2.3) and (2.5) which is identical to our scaling rule (5.6) except that it is suppressed by an overall factor of M^2/m_p^2 for $d > 2$, where d is the number of derivatives. They say in their footnote 2 that this rule comes from integrating out a PNGB field of mass M. However we find that the detailed power counting calculations given in the second example in their Sec. 2.2 actually yield our scaling rule rather than theirs.

an adiabatic approximation. At zeroth order in $1/M$, the theory obtained for the fields χ^a is a nonlinear sigma model where the potential is just the potential of the action (F.2) evaluated on the surface $\psi^\Gamma = 0$, and the target space metric is just the metric induced on the surface from the metric q_{AB}.

To obtain the higher order corrections we can proceed as follows. The equation of motion for the fields ψ^Γ is

$$\Box\psi^\Sigma + \Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b + \Gamma^\Sigma_{\Theta\Upsilon}\vec{\nabla}\psi^\Theta \cdot \vec{\nabla}\psi^\Upsilon + 2\Gamma^\Sigma_{a\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\psi^\Theta = H_0^2 q^{\Sigma a}V_{,a} + H_0 M q^{\Sigma\Theta}V_{,\Theta}. \tag{F.4}$$

Here the connection coefficients are those of the target space metric q_{AB}. We next expand this equation to linear order in ψ^Γ and use the condition (F.3) to obtain

$$\begin{aligned} \Box\psi^\Sigma + \left[\Gamma^\Sigma_{ab,\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b - H_0^2 q^{\Sigma a}{}_{,\Theta}V_{,a} - M^2 q^{\Sigma\Upsilon}V_{,\Upsilon\Theta}\right]\psi^\Theta \\ +2\Gamma^\Sigma_{a\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\psi^\Theta = -\Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b + H_0^2 q^{\Sigma a}V_{,a}, \end{aligned} \tag{F.5}$$

where all the metric coefficients, connection coefficients and their derivatives are evaluated at $\psi^\Gamma = 0$. Now in the large M or adiabatic limit, the dominant term on the left hand side will be the term proportional to M^2, and dropping the other terms gives a simple algebraic equation for the leading order contribution to ψ^Γ:

$$\left[q^{\Sigma\Upsilon}V_{,\Upsilon\Theta}\right]\psi^\Theta = \frac{1}{M^2}\left[\Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b - H_0^2 q^{\Sigma a}V_{,a}\right]. \tag{F.6}$$

Substituting the solution given by Eq. (F.6) into the action (F.2) gives the required, $O(1/M^2)$ corrections to the action. The first term on the right hand side of Eq. (F.6) will give nonlinear corrections to the kinetic energy. (We assume that the second fundamental form or extrinsic curvature of the surface $\psi^\Gamma = 0$ is nonzero, otherwise these corrections would vanish.)

As a simple example, consider the theory

$$\mathcal{L} = -\frac{1}{2}(\nabla\chi)^2 - \frac{1}{2}(\nabla\psi)^2 - \frac{1}{2}M^2\psi^2 + \psi(\nabla\chi)^2/m_p. \tag{F.7}$$

The equation of motion for ψ is $\Box\psi - M^2\psi = (\nabla\chi)^2/m_p$ with leading order solution $\psi = -(\nabla\chi)^2/(m_p M^2)$. The corresponding corrections to the action for χ scale as $(\nabla\chi)^4/(m_p^2 M^2)$, in agreement with Eq. (5.6). The scaling (5.6) of other operators can be derived similarly.